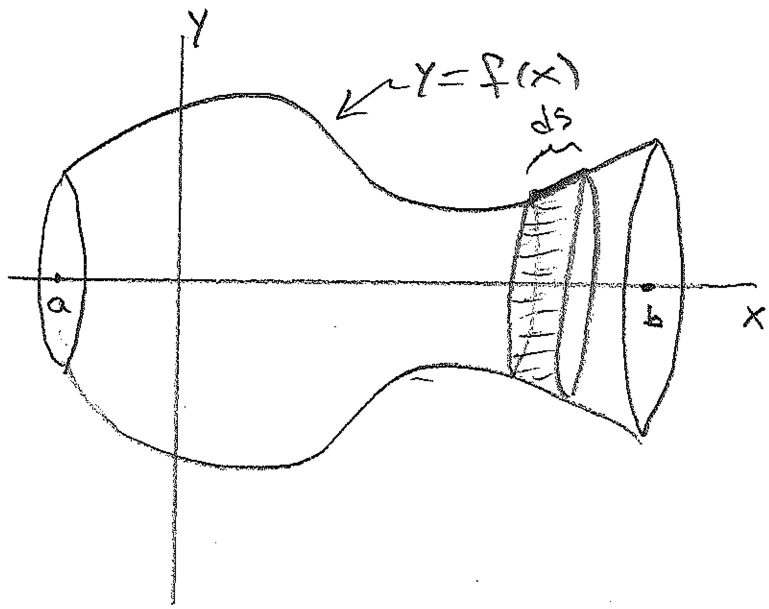
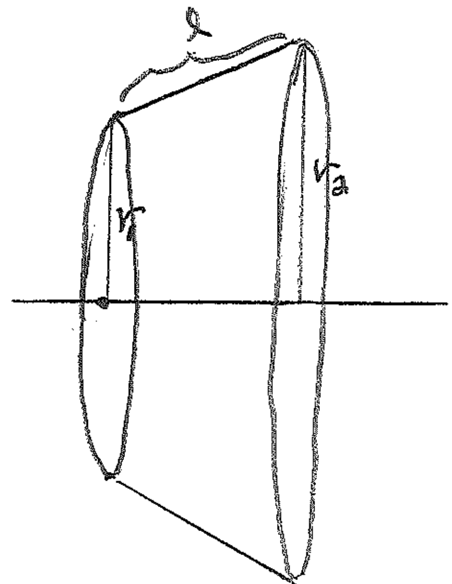
Area of surface of revolution



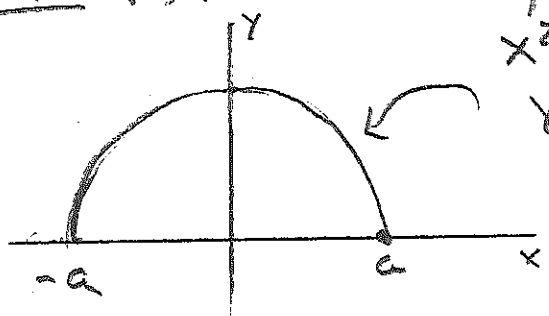


$$A = 2\pi\left(\frac{r_1+r_2}{2}\right)l$$

$$dA = 2\pi r\, ds \qquad r = f(x)$$

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\, dx$$

__Ex.__ Area of a sphere of radius a



$$x^2+y^2=a^2$$

$$y=\sqrt{a^2-x^2} \qquad r = y$$

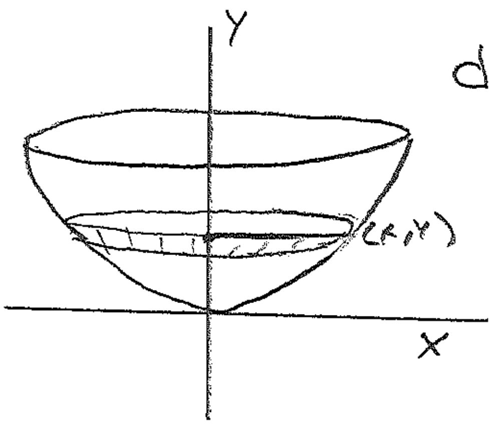
$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)^2$$

$$= 1+\frac{x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$$

$$A = \int_{-a}^{a} 2\pi\sqrt{a^2-x^2}\left(\frac{a}{\sqrt{a^2-x^2}}\right)dx = \int_{-a}^{a} 2\pi a\, dx$$

$$= 2\pi a(2a) = 4\pi a^2$$

Ex The arc of the parabola $y = x^2$, $0 \le x \le 1$ is rotated about the y-axis. Find the area of the surface of revolution.



$r = x$

$dA = 2\pi x \, ds$

$$A = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_0^1 2\pi x \sqrt{1 + 4x^2} \, dx$$

$u = 1 + 4x^2$

$du = 8x \, dx$

$$= \int_1^5 2\pi \sqrt{u} \, \frac{du}{8}$$

$$A = \frac{\pi}{4} \int_1^5 u^{1/2} du = \frac{\pi}{4} \frac{2}{3} u^{3/2} \Big]_1^5$$

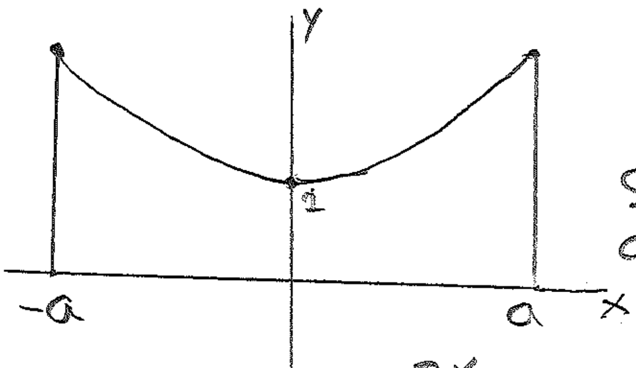
$$= \frac{\pi}{6}(5^{3/2} - 1) = \frac{\pi}{6}(5\sqrt{5} - 1)$$

Thus $\lim_{\|P\| \to 0} \sum_{i=1}^{N} l_i = \int_a^b \sqrt{1+f'(x)^2}\,dx$ so

continuous

$$S = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

<u>Ex.</u> Catenary - Hanging chain



$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \qquad e^x e^{-x} = 1$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{e^{2x} + 2\cancel{e^x e^{-x}} + e^{-2x}}{4}$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 = \cosh^2(x)$$

$$S = \int_{-a}^{a} \cosh(x)\,dx = \sinh(x)\Big]_{-a}^{a}$$

$$= \sinh(a) - \sinh(-a) = 2\sinh(a) = e^a - e^{-a}$$

Note: Have shown $\cosh^2(x) - \sinh^2(x) = 1$

Integral holds since

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$