Taylor Series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

What happens when $c=0$?

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{(Maclaurin Series)}$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Find the Maclaurin series and the interval on which it is valid.

$f(x) = \cos(3x)$

We know $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

so $\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 9^n \cdot x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-9)^n x^{2n}}{(2n)!}$

$$= 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} \dots$$

Find the Maclaurin series for $f(x) = \cos(x^3)$ and use it to determine $f^{(6)}(0)$.

Using knowledge from above:

$$\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots$$

So by expansion, the coefficient of $x^6 = \frac{-1}{2!}$

But the Maclaurin series definition says
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ the coefficient of $x^6$

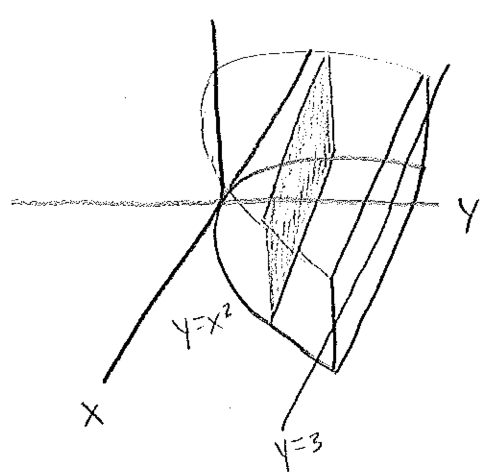
is $\frac{f^{(6)}(0)}{6!}$

So $\frac{-1}{2!} = \frac{f^{(6)}(0)}{6!}$

Solving, we get $f^{(6)}(0) = \frac{-(6!)}{2!} = \frac{-6\cdot 5\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{2}\cdot \cancel{1}}$

$= -6\cdot 5\cdot 4\cdot 3 = 972$

Find the volume of the solid whose base is the region enclosed by $y=x^2$ and $y=3$ and whose cross-sections perpendicular to the y-axis are squares.




$A = s^2$
$= (2\sqrt{y})^2$
$= 4y$

$y = x^2$
$x = \sqrt{y}$
$s = 2x = 2\sqrt{y}$

$V = \int_0^3 4y\,dy = \frac{4y^2}{2}\Big|_0^3 = 2y^2\Big|_0^3$

$= 2(3)^2 - 2(0)^2$

$= 18$

Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance r cm from the center is $v(r) = (16 - r^2)$ cm/s.

$$\text{Flow rate } Q = 2\pi \int_0^R r\,v(r)\,dr$$

$$Q = 2\pi \int_0^4 r(16 - r^2)\,dr = 2\pi \int_0^4 (16r - r^3)\,dr$$

$$= 2\pi \left[\frac{16r^2}{2} - \frac{r^4}{4}\right]\Big|_0^4 = 2\pi\left[8r^2 - \frac{r^4}{4}\right]\Big|_0^4$$

$$= 2\pi\left(8(4)^2 - \frac{4^4}{4}\right) - 2\pi\left(8(0)^2 - \frac{0^4}{4}\right)$$

$$= 2\pi(64) = 128\pi \text{ cm}^3/\text{s}$$

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Find the average speed over the time interval $[1, 5]$ of a particle whose position at time $t$ is $s(t) = t^3 - 6t^2$ m/s.

$$s(t) = t^3 - 6t^2$$

$$\text{velocity} = s'(t) = 3t^2 - 12t$$

$$= t(3t - 12)$$

$$\text{velocity} = 0 \text{ when } t = 0, t = 4$$

we can see that $3t^2 - 12t$ is negative on $[1, 4)$ and positive on $(4, 5]$

Average velocity

$$\frac{1}{5-1}\int_1^5 |3t^2 - 12t|\,dt = \frac{1}{4}\int_1^4 (12t - 3t^2)\,dt + \frac{1}{4}\int_4^5 (3t^2 - 12t)\,dt$$

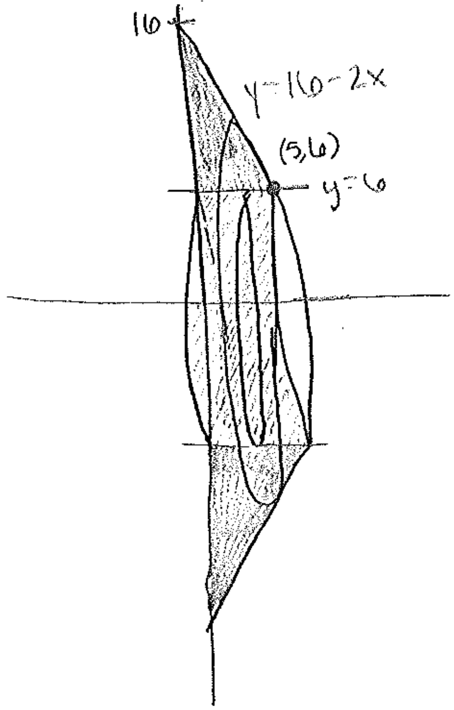
$$= \frac{1}{4}\left[6t^2 - t^3\right]\Big|_1^4 + \frac{1}{4}\left[t^3 - 6t^2\right]\Big|_4^5$$

$= 1/4[6(4)^2-(4)^3]-1/4[6(1)^2-(1)^3]+1/4[(5)^3-6(5)^2]-1/4[(4)^3-6(4)^2]$

$= 1/4(32)-1/4(5)+1/4(-25)-1/4(-32)$

$= 17/2$ m/s

Find the volume of the solid obtained by rotating the region enclosed by the graphs of $y=16-2x$, $y=6$, and $x=0$ about the x-axis.




$$V=\pi\int_0^5[(16-2x)^2-6^2]\,dx$$

$$=\pi\int_0^5(256-64x+4x^2-36)\,dx$$

$$=\pi\int_0^5(4x^2-64x+220)\,dx$$

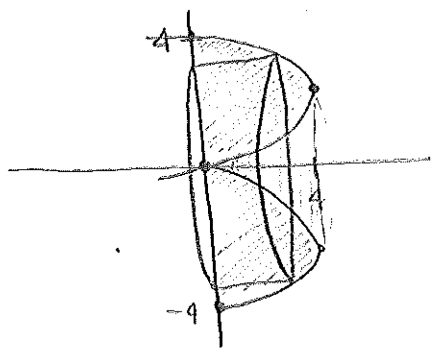
$$=\pi\left[\frac{4x^3}{3}-32x^2+220x\right]\Big|_0^5$$

$$=\pi\left[\frac{4(5)^3}{3}-32(5)^2+220(5)\right]$$

$$=\pi\left[\frac{500}{3}-800+1100\right]$$

$$=\frac{1400\pi}{3}$$

Sketch the region enclosed by $x = y(4-y)$ and $y=0$ and use the shell Method to calculate the volume of rotation about the x-axis.




$$V = 2\pi \int_0^4 y\left(y(4-y)\right)dy$$

$$= 2\pi \int_0^4 y(4y-y^2)dy$$

$$= 2\pi \int_0^4 (4y^2 - y^3)dy$$

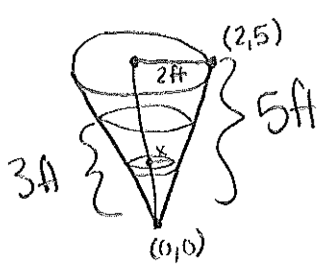
$$= 2\pi \left[\frac{4y^3}{3} - \frac{y^4}{4}\right]\Big|_0^4$$

$$= 2\pi\left[\frac{4(4)^3}{3} - \frac{(4)^4}{4}\right] - 2\pi\left[\frac{4(0)^3}{3} - \frac{(0)^4}{4}\right]$$

$$= 2\pi\left[\frac{256}{3} - \frac{256}{4}\right]$$

$$= 2\pi\left[64/3\right] = 128\pi/3$$

Find the work required to pump 3 ft of water out of a tank in the shape of an inverted circular cone that is 5ft tall and has radius 2ft.




$$y = \frac{5}{2}x \qquad x = \frac{2}{5}y$$

$$A(y) = \pi r^2 = \pi x^2 = \pi\left(\frac{2}{5}y\right)^2 = \frac{4\pi}{25}y^2$$

$$dW = F \cdot d$$
$$= m \cdot a \cdot d$$
$$= \rho \cdot V \cdot a \cdot d$$

$V = A(y)\,dy = \frac{4\pi}{25} y^2 dy$

So $dW = 1000 \cdot \frac{4\pi}{25} y^2 \cdot 9.8 \cdot (5-y) \cdot dy$

$$W = 1000 \cdot 9.8 \cdot \frac{4\pi}{25} \int_0^3 y^2 (5-y)\,dy$$

$$= 1568\pi \int_0^3 (5y^2 - y^3)\,dy$$

$$= 1568\pi \left[ \frac{5y^3}{3} - \frac{y^4}{4} \right] \Big|_0^3$$

$$= 1568\pi \left[ \frac{5(3)^3}{3} - \frac{(3)^4}{4} \right]$$

$$= 1568\pi \left[ 45 - \frac{81}{4} \right]$$

$$= 38{,}808 \text{ ft}^3$$

Use integration by parts to integrate $\int x^2 \ln x \, dx$

$$\int x^2 \ln x \, dx$$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$$\int x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

Use integration by parts to integrate $\int x^2 \sin x \, dx$

$u = x^2 \qquad dv = \sin x \, dx$

$du = 2x \, dx \qquad v = -\cos x$

$$\int x^2 \sin x \, dx = -x^2\cos x - \int -2x\cos x \, dx = -x^2 \cos x + \int 2x\cos x \, dx$$

$u = 2x \qquad dv = \cos x \, dx$

$du = 2dx \qquad v = \sin x$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x \, dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Evaluate $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$

$$\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \int_0^{\pi/2} \sin^2 x \cos^2 x \, (\cos x) \, dx$$

$$= \int_0^{\pi/2} (\sin^2 x)(1 - \sin^2 x) \cos x \, dx = \int_0^1 u^2 (1 - u^2) \, du$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int_0^1 (u^2 - u^4) \, du$$

$$= \left[ \frac{u^3}{3} - \frac{u^5}{5} \right] \Bigg|_0^1$$

$\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

Use a trig substitution to evaluate $\int \frac{dy}{y^2 \sqrt{5 - y^2}}$

Substitution: $y = \sqrt{5} \sin\theta$

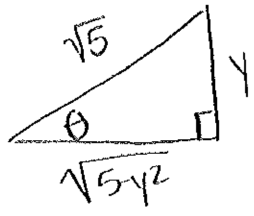
$dy = \sqrt{5} \cos\theta \, d\theta$

$\sqrt{5 - y^2} = \sqrt{5 - (\sqrt{5}\sin\theta)^2} = \sqrt{5 - 5\sin^2\theta} = \sqrt{5(1 - \sin^2\theta)}$

$= \sqrt{5\cos^2\theta} = \sqrt{5} \cos\theta$

$$\int \frac{dy}{y^2 \sqrt{5 - y^2}} = \int \frac{\sqrt{5} \cos\theta \, d\theta}{(\sqrt{5}\sin\theta)^2 \cdot \sqrt{5}\cos\theta} = \int \frac{d\theta}{5\sin^2\theta} = \frac{1}{5} \int \csc^2\theta \, d\theta$$

$$= \frac{1}{5}(-\cot\theta) + C$$




$$= -\frac{1}{5} \cot\theta + C = -\frac{1}{5} \left( \frac{\sqrt{5 - y^2}}{y} \right) + C$$