#1

$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ . Try the Root Test, noting that the terms are positive (the Ratio Test also works).

$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}=\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$ . Note that this is an indeterminate form $1^{\infty}$ . So,

$\lim_{n\to\infty}\ln\left(\left(\frac{n}{n+1}\right)^{n}\right)=\lim_{n\to\infty}n\cdot\ln\left(\frac{n}{n+1}\right)=\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}=LR=\lim_{n\to\infty}\frac{\left(\frac{n+1}{n}\right)\cdot\frac{1}{(n+1)^2}}{-\frac{1}{n^2}}=\lim_{n\to\infty}-\frac{n^2(n+1)}{n(n+1)^2}=-1\cdot$

So, $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=e^{-1}=\frac{1}{e}<1$ . So, the series converges (absolutely) by the Root Test.

#2

The series $\sum_{n=1}^{\infty}(-1)^{n}2^{\frac{1}{n}}$ diverges by the Test for Divergence. $\lim_{n\to\infty}(-1)^{n}2^{\frac{1}{n}}$ does not exist. The terms do not approach zero. The terms alternate between numbers close to 1 and -1, making the series bat back and forth.

#3

$\sum_{n=1}^{\infty}\frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}$ . Note that $\sin\left(\frac{1}{n}\right)<\frac{1}{n}$ for $n>1$ . So, $\frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}<\frac{\frac{1}{n}}{n^{\frac{1}{2}}}=\frac{1}{n^{\frac{3}{2}}}$ and $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges (P-series, p>1).

So, the original series converges by the Comparison Test.

#4

$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ . Note that $(\ln n)^{\ln n}=\left(e^{\ln\ln n}\right)^{\ln n}=\left(e^{\ln n}\right)^{\ln\ln n}=n^{\ln\ln n}$ . Since $\ln\ln n\to\infty$ as $n\to\infty$ , $\ln\ln n>2$ for $n$ large.

So, $(\ln n)^{\ln n}>n^2$ and $\frac{1}{(\ln n)^{\ln n}}<\frac{1}{n^2}$ . Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ converges (p-series, p=2), so does $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

#5

$\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ . Use the Ratio Test.

$\lim_{k\to\infty}\left|\frac{\frac{k+1+5}{5^{k+1}}}{\frac{k+5}{5^k}}\right|$ Invert and multiply the bottom fraction to cancel the $5^k$ .

$\lim_{k\to\infty}\left|\frac{k+6}{5(k+5)}\right| = \lim_{k\to\infty}\left|\frac{k+6}{5k+25}\right|$

Divide the numerator and the denominator by k to get $\lim_{k\to\infty}\left|\frac{1+\frac{6}{k}}{5+\frac{25}{k}}\right|=\frac{1}{5}.$

The limit approaches $\frac{1}{5}$ as $k\to\infty$ . The limit is less than 1, so $\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ converges absolutely by the Ratio Test.

#6

$\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^{n}$

Apply the Root Test. $\lim_{n\to\infty}\sqrt[n]{(\sqrt[n]{2}-1)^{n}}=\lim_{n\to\infty}\sqrt[n]{2}-1=\lim_{n\to\infty}2^{\frac{1}{n}}-1=1-1=0<1.$

So, $\sum_{n=1}^{\infty}(\sqrt[n]{2}-1)^{n}$ converges absolutely (thus, converges) by the Root Test.

#7

$\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ . Note that, since $(k+1)^3>k^3$ , $\frac{k\ln k}{(k+1)^3}<\frac{k\ln k}{k^3}=\frac{\ln k}{k^2}$ . Now, consider $\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$ . Try the Integral Test. $f(x)=\frac{\ln x}{x^2}$ is positive, decreasing, continuous, and agrees with the terms of the sequence. Now, $\int_1^{\infty}\frac{\ln x}{x^2}dx$ may be evaluated by Integration by Parts. Let $u=\ln x$ and $dv=\frac{1}{x^2}dx$ . Then, $du=\frac{1}{x}dx$ and $v=-\frac{1}{x}$ .

So, $\int_1^{\infty}\frac{\ln x}{x^2}dx=\lim_{R\to\infty}\int_1^{R}\frac{\ln x}{x^2}dx=\lim_{R\to\infty}\left[-\frac{\ln x}{x}\Big|_1^R-\int_1^R-\frac{\ln x}{x}dx\right]$ . Now, use substitution for the integral to give:

$$\int_1^{\infty}\frac{\ln x}{x^2}dx=\lim_{R\to\infty}\int_1^{R}\frac{\ln x}{x^2}dx=\lim_{R\to\infty}\left[-\frac{\ln x}{x}\Big|_1^R+\int_0^{\ln R}udu\right]=\lim_{R\to\infty}\left[-\frac{\ln R}{R}+\frac{1}{2}u^2\Big|_0^{\ln R}\right]=\lim_{R\to\infty}\left[-\frac{\ln R}{R}+\frac{1}{2}(\ln R)^2\right]=\lim_{R\to\infty}\frac{(\ln R)^2-2\frac{\ln R}{R}}{2}=\infty$$

. So, the improper integral diverges. So, the series diverges.

#8

$\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$

It is alot like a p-series. You can compare it (using the Limit Comparison Test) to the divergent Harmonic Series $\sum_{n=1}^{\infty}\frac{1}{n}$.

(You might think to do this because when you go out really far the + 1 does not effect the expression very much at all.)

By the Limit Comparison Test, $\lim_{n\to\infty}\frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n(n^2+1)}{n^3+1}=\lim_{n\to\infty}\frac{n^3+n}{n^3+1}=1>0$. So, the original series diverges also.

#9

$\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ . Use the Limit Comparison test with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ , which converges (p-series, p=2).

$\lim_{n\to\infty}\frac{\frac{\sqrt{n^2-1}}{n^3+2n^2+5}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5}=\lim_{n\to\infty}\frac{\sqrt{n^6-n^4}}{n^3+2n^2+5}$ . Dividing numerator and denominator by $n^3$ gives $\lim_{n\to\infty}\frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{2}{n}+\frac{5}{n^3}}=1>0.$

So, by the Limit Comparison Test, the original series converges as well.

#10

$\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

Use the Integral Test.

Note that $f(x)=\frac{1}{x\sqrt{\ln(x)}}$ is a continuous, positive, decreasing function on $[2,\infty)$ . Substitute $u=\ln(x)$

$$\int_2^{\infty}\left(\frac{dx}{x\sqrt{\ln(x)}}\right)=\int_{\ln(2)}^{\infty}\left(\frac{du}{\sqrt{u}}\right)=\lim_{R\to\infty}-2\sqrt{u}\Big|_{\ln(2)}^R=\lim_{R\to\infty}\left(-2\sqrt{R}+2\sqrt{\ln(2)}\right)=2\sqrt{\ln(2)}$$

Since $\int_2^{\infty}f(x)dx$ diverges, $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ also diverges.

#11

$\sum_{n=1}^{\infty}\frac{\sin(2n)}{1+2^n}$ . Try to show absolute convergence. Note that $|\sin(2n)|\le 1$ for all $n$ . So, $\left|\frac{\sin(2n)}{1+2^n}\right|\le\frac{1}{1+2^n}<\frac{1}{2^n}$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges (geometric, $|r|<1$ .) So, the series converges absolutely (hence, converges).

#12

$\sum_{n=1}^{\infty}\frac{4^k}{2^k+3^k}$ Compare it to a geometric series.

$2^k+3^k<3^k+3^k$ and so $\frac{4^k}{2^k+3^k}>\frac{4^k}{3^k+3^k}=\frac{4^k}{2\cdot 3^k}=\frac{4}{6}\cdot\left(\frac{4}{3}\right)^{k-1}=\frac{2}{3}\cdot\left(\frac{4}{3}\right)^{k-1}$

Since $\sum_{k=1}^{\infty}\frac{2}{3}\cdot\left(\frac{4}{3}\right)^{k-1}$ diverges (geometric series, with $r>1$), the original series also diverges by the Comparison Test.

#13

$\sum_{n=1}^{\infty}$ sin(n). Use the Test for Divergence.

$\lim_{n\to\infty}$ sin(n) does not exist so $\sum_{n=1}^{\infty}$ sin(n) diverges.

#14

$\sum_{k=1}^{\infty}\frac{2^k}{k!}$

Use the Ratio Test!

$$\lim_{k\to\infty}\frac{\frac{2^{k+1}}{(k+1)!}}{\frac{2^k}{k!}}=\lim_{k\to\infty}\frac{(2^{k+1})(k!)}{(k+1)!(2^k)}=\lim_{k\to\infty}\frac{(2^{k+1})(k!)}{(k+1)k!(2^k)}=\lim_{k\to\infty}\frac{2^{k+1}}{(k+1)(2^k)}=\lim_{k\to\infty}\frac{2}{k+1}=0<1$$

By the Ratio Test, the series converges absolutely (thus, converges).

#15

$\sum_{n=1}^{\infty}\frac{2^k\cdot k!}{(k+2)!}$

Apply the Ratio Test. Note that the terms are positive.

$$\lim_{k\to\infty}\frac{\frac{2^{k+1}\cdot(k+1)!}{(k+3)!}}{\frac{2^k\cdot k!}{(k+2)!}}=\lim_{k\to\infty}\frac{2^{k+1}\cdot(k+1)!}{(k+3)!}\cdot\frac{(k+2)!}{2^k\cdot k!}=\lim_{k\to\infty}\frac{2(k+1)}{k+3}=2>1.$$

So, by the Ratio Test, the series is divergent.

#16

$\sum_{n=1}^{\infty}\frac{1}{n+4^n}$

$n+4^n>4^n$ for $n\geq 1$ . So, $0<\frac{1}{n+4^n}<\frac{1}{4^n}$.

$\sum_{n=1}^{\infty}\frac{1}{4^n}$ = $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ = $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)^{n-1}$

This series is a convergent geometric series. So, by the Comparison Test, the other series converges also.

#17

$\sum_{n=1}^{\infty}\frac{4^n\cdot n^3}{n!}$

Apply the Ratio Test. Terms are positive so no need for absolute value sign.

$$\lim_{x\to\infty}\frac{\frac{4^{n+1}\cdot(n+1)^3}{(n+1)!}}{\frac{4^n\cdot n^3}{n!}}=\lim_{x\to\infty}\frac{(n!)\cdot 4^{n+1}\cdot(n+1)^3}{(n+1)!\cdot 4^n\cdot n^3}=\lim_{x\to\infty}\left(\frac{n!}{(n+1)\cdot n!}\right)\cdot\left(\frac{4^n\cdot 4}{4^n}\right)\cdot\left(\frac{(n+1)^3}{n^3}\right)$$

After canceling like terms, we have:

$$4\cdot\lim_{x\to\infty}\frac{(n+1)^2}{n^3}=4\cdot\lim_{x\to\infty}\frac{n^2+2n+2}{n^3}\cdot\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)=4\cdot\lim_{x\to\infty}\frac{\left(\frac{1}{n}\right)+\left(\frac{2}{n^2}\right)+\left(\frac{2}{n^3}\right)}{1}=0<1$$

So, $\sum_{n=1}^{\infty}\frac{4^n\cdot n^3}{n!}$ absolutely converges (thus converges) by the Ratio Test.

#18

$\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty}\left(\frac{(-2)^2}{n}\right)^n = \sum_{n=1}^{\infty}\left(\frac{4}{n}\right)^n$ Apply Root Test.

$\lim_{n\to\infty}\sqrt[n]{\left(\frac{4}{n}\right)^n}$. The root and exponent cancel which leaves $\lim_{x\to\infty}\left(\frac{4}{n}\right) = 0 < 1$.

So, the series converges absolutely by the Root Test.