Series Toolbox

- **Test for Divergence (The Hammer):** If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. That is to say, if the terms of the series are not getting smaller, then the series has no chance to converge.

  Remember that the converse is not true. That is, if $\lim_{n\to\infty} a_n = 0$ then the series only has a *chance* to converge, and another test must be used.

- **Geometric Series:** A geometric series in the form $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ will diverge if $|r| \geq 1$. If $|r| < 1$, the series will converge and

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- **Integral Test:** Suppose f is continuous, positive, and monotone decreasing on $[1, \infty)$ and $f(n) = a_n$ for all positive integers n from some point on. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_{n=1}^{\infty} f(x)\, dx$ converges.

- **P-Series:** A series of the form $\sum_{n=1}^{\infty} {}^{1}/_{n^p}$ will converge if $p > 1$ and diverge if $p \leq 1$. Note that the special case for $p = 1$ is the Harmonic Series, which famously diverges.

- **Comparison Test:** Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
  - (i) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n from some point on, then $\sum_{n=1}^{\infty} a_n$ also converges.
  - (ii) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all n from some point on, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **Limit Comparison Test:** Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive term series and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

  If $0 < L < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

- **Alternating Series Test:** If $\{a_n\}$ is a positive sequence satisfying
  - (i) $a_n \geq a_{n+1}$ for all n from some point on, and
  - (ii) $\lim_{n\to\infty} a_n = 0$,

  then the alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

- **Alternating Series Estimation Theorem:** If the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges to S, then

$$|S - S_N| < a_{N+1},$$

  where S_N is the Nth partial sum of the series. That is to say, the error in the Nth partial sum is less than the $(N+1)$st term of the series.

- **Ratio Test:**
  (i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.
  (ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.
  (iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive.

- **Root Test:**
  (i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.
  (ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.
  (iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then the test is inconclusive.

- **Telescoping Series:** Telescoping series are positive term series in which the general partial sums are easy to compute due to cancellation between the terms. This is best demonstrated with an example. Suppose we wish to investigate the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

First we can use partial fraction decomposition to rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Then the first few partial sums are:

$$\begin{aligned} S_1 &= \left(\frac{1}{1} - \frac{1}{2}\right) = 1 - \frac{1}{2} \\ S_2 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\ S_3 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} \end{aligned}$$

In general, the Nth partial sum is

$$S_N = 1 - \frac{1}{N+1}.$$

Since a series converges if and only if its sequence of partial sums converges, and

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

we conclude that the series converges to 1.