Math 114 Worksheet 10

Taylor Series and Taylor Polynomials

1. Find a power series representation for

(a) $f(x) = x\cos(x^2)$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$$

$$x\cos(x^2) = x\sum_{n=0}^{\infty} (-1)^n \frac{(x)^{4n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!} = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \ldots$$

(b) $g(x) = (1+x)e^{-x} = e^{-x} + xe^{-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots, \quad xe^{-x} = x\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!} = x - x^2 + \frac{x^3}{2!} - \ldots$$

$$g(x) = e^{-x} + xe^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{x^n + x^{n+1}}{n!}\right)$$

2. Show that $\lim_{x\to 0} \frac{e^x - \cos(x)}{\sin(x)} = 1$ using power series. Verify your answer with l'Hospital's Rule.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\lim_{x\to 0} \frac{(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) - (\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots)}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots} =$$

$$\lim_{x\to 0} \frac{\cancel{x}(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots + \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \ldots)}{\cancel{x}(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots)} =$$

$$\frac{1 + 0 + 0 + \ldots + 0 - 0 + 0 - \ldots}{1 - 0 + 0 - 0 + \ldots} = \frac{1}{1} = 1$$

L'Hopital's rule yields:

$$\lim_{x\to 0} \frac{e^x + \sin(x)}{\cos(x)} = \frac{e^0 + \sin(0)}{\cos(0)} = \frac{1+0}{1} = 1.$$

3. What is $T_3(x)$ centered at $a=3$ for a function $f(x)$ where $f(3)=9$, $f'(3)=8$, $f''(3)=4$, and $f'''(3)=12$?

$$T_3(x) = f(3)+f'(3)(x-3)+f''(3)(x-3)^2+f'''(3)(x-3)^3$$
$$= 9+8(x-3)+\frac{4}{2!}(x-3)^2+\frac{12}{3!}(x-3)^3$$

4. Calculate the Taylor polynomials $T_2(x)$ and $T_3(x)$ centered at $x=a$ for the given function and value of a.

(a) $f(x)=\tan x$, $a=\pi/4$ $\qquad f(\frac{\pi}{4})=1$

$f'(x)=\sec^2 x$ $\qquad f'(\frac{\pi}{4})=2$

$f''(x)=2\sec^2 x\tan x$ $\qquad f''(\frac{\pi}{4})=4$

$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x$ $\qquad f'''(\frac{\pi}{4})=16$

$T_2(x)=1+2(x-\frac{\pi}{4})+4(x-\pi/4)^2$

$T_3(x)=1+2(x-\pi/4)+4(x-\pi/4)^2+16(x-\pi/4)^3$

(b) $f(x)=x^2e^{-x}$, $a=1$ $\qquad f(1)=\frac{1}{e}$

$f'(x)=2xe^{-x}-x^2e^{-x}$ $\qquad f'(1)=\frac{2}{e}-\frac{1}{e}=\frac{1}{e}$

$f''(x)=2e^{-x}-2xe^{-x}-2xe^{-x}+x^2e^{-x}$

$=2e^{-x}-4xe^{-x}+x^2e^{-x}$ $\qquad f''(1)=\frac{2}{e}-\frac{4}{e}+\frac{1}{e}=\frac{-1}{e}$

$f'''(x)=2e^{-x}-4e^{-x}+4xe^{-x}+2xe^{-x}-x^2e^{-x}$

$=-2e^{-x}+6xe^{-x}-x^2e^{-x}$ $\qquad f'''(1)=\frac{-2}{e}+\frac{6}{e}-\frac{1}{e}=\frac{3}{e}$

$T_2(x)=\frac{1}{e}+\frac{1}{e}(x-1)-\frac{1}{e}(x-1)^2$

$T_3(x)=$ 〃 $+\ ^3/_e(x-1)^3$

(c) $f(x)=\frac{\ln x}{x}$, $a=1$ $\qquad f(1)=0$

$f'(x)=\frac{1-\ln(x)}{x^2}$ $\qquad f'(1)=1$

$f''(x)=\frac{-3+2\ln(x)}{x^3}$ $\qquad f''(1)=\frac{-1-2}{1}=-3$

$f'''(x)=\frac{x^4(\frac{2}{x})-(-3+2\ln(x))3x^2}{x^6}$

$$= \frac{2x^3 + 9x^2 - 6x^2\ln(x)}{x^6}$$

$$= \frac{2x + 9 - 6\ln(x)}{x^4} \qquad f'''(1) = 11$$

$T_2(x) = (x-1) - 3(x-1)^2$

$T_3(x) = (x-1) - 3(x-1)^2 + 11(x-1)^3$

5. Let $T_2(x)$ be the Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 4$. Apply the error bound to find the maximum possible value of $|f(1.1) - T_2(1.1)|$. Show that we can take $K = e^{1.1}$.

	$f(4) = 2$
$f'(x) = \frac{1}{2}x^{-1/2}$	$f'(4) = \frac{1}{4}$
$f''(x) = -\frac{1}{4}x^{-3/2}$	$f''(4) = -\frac{1}{4}\left(\frac{1}{2^3}\right) = -\frac{1}{32}$
$f'''(x) = \frac{3}{8}x^{-5/2}$	$f'''(4) = \frac{3}{8}\left(\frac{1}{2^5}\right) = 3/256$

$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2$

$T_2(1.1) = 2 + \frac{1}{4}(-2.9) - \frac{1}{32}(-2.9)^2$

$|f(1.1) - T_2(1.1)| < \left|3/256\,(-2.9)^3\right| = .2858$

$|f^{(k)}(x)| \leq e^{1.1}$

6.(a) Let $f(x) = 3x^3 + 2x^2 - x - 4$. Calculate $T_k(x)$ for $k = 1, 2, 3, 4, 5$ at both $a = 0$ and $a = 1$. Show that $T_3(x) = f(x)$ in both cases.

$f(x) = 3x^3 + 2x^2 - x - 4$	$f(0) = -4$	$f(1) = 0$
$f'(x) = 9x^2 + 4x - 1$	$f(0) = -1$	$f(1) = 12$
$f''(x) = 18x + 4$	$f(0) = 4$	$f(1) = 22$
$f'''(x) = 18$	$f(0) = 18$	$f(1) = 18$
$f''''(x) = 0$	$f(0) = 0$	$f(1) = 0$
$f^{(5)}(x) = 0$	$f(0) = 0$	$f(1) = 0$

$T_1(x) = -4 - x \qquad T_1(x) = 12(x-1) = 12x - 12$

$T_2(x) = -4 - x + \frac{4x^2}{2!} \qquad T_2(x) = 12x - 12 + 11(x-1)^2$

$\qquad\qquad\qquad\qquad = 11x^2 - 10x - 1$

$$T_3(x) = -4 - x + \frac{4x^2}{2!} + \frac{18x^3}{3!} \qquad T_3(x) = 11x^2 - 10x - 1 + 3(x-1)^3$$

$$= -4 - x + 2x^2 + 3x^3 \qquad = 3x^3 + 2x^2 - x + 4.$$

$T_4(x) = 11 \qquad T_4(x) = 11$

$T_5(x) = 11 \qquad T_5(x) = 11$

(b) Let $T_n(x)$ be the n^{th} Taylor polynomial at $x=a$ for a polynomial $f(x)$ of degree n. Based on part (a), guess the value of $|f(x) - T_n(x)|$. Prove that your guess is correct using the error bound.

$|f(x) - T_n(x)| = 0$, this guess is correct because $|f(x) - T_n(x)| < T_{n+1}(x) - T_n(x)$ and since $f^{(n+1)}(x) = 0$, $T_{n+1}(x) - T_n(x) = 0$.