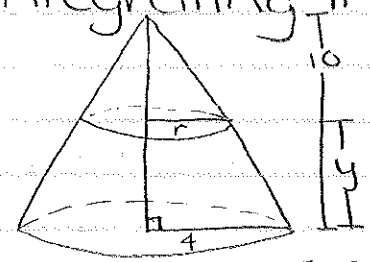
Math 114: Worksheet 11

Volumes of Solids with Known Cross Sections

1. Conceptual Understanding: If a solid has a cross-sectional area given by the function $A(x)$, which integral should be evaluated to find the volume of the solid?

$\int A(x)\,dx$

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4. Use similar triangles to find the area of a horizontal cross section at a height y. Using this area, calculate V by integrating the cross-sectional area.



Similar triangles have similar ratios, so

$\frac{4}{r} = \frac{10}{10-y}$

$r = 4 - \frac{2}{5}y$

The cross-sections of a circular cone are circles and $A_o = \pi r^2$. So $\int_0^{10} \pi(4 - \frac{2}{5}y)^2\,dy$. $-\frac{2}{5}(4) + -\frac{2}{5}(4)$

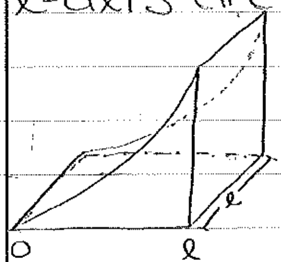
$= \pi \int_0^{10} 16 - \frac{16}{5}y + \frac{4}{25}y^2\,dy$

$= \pi\left(16y - \frac{8}{5}y^2 + \frac{4}{75}y^3\right)\Big|_0^{10}$

$= \pi\left(160 - 160 + \frac{4}{3}(40)\right) = \pi\left(\frac{160}{3}\right) = \frac{160\pi}{3}.$

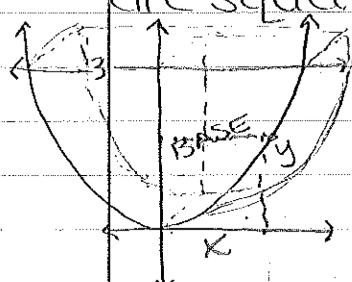
We can check $V_\Delta = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(16)10 = \frac{160\pi}{3}$ ✓.

3. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval [0, ℓ] along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.



$$\int_0^{\ell} \ell(x^2)\,dx = \frac{1}{3}\ell x^3\Big|_0^{\ell}$$
$$= \frac{1}{3}\ell(\ell)^3$$
$$= \frac{1}{3}\ell^4$$

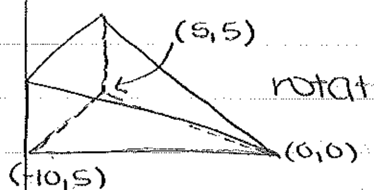
4. Calculate the volume of the following solid. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.



integrating with respect to y, so the area of the circles need to be in terms of y.
So $x = 2\sqrt{y}$

$$\int_0^3 4y\,dy = 2y^2\Big|_0^3 = 18$$

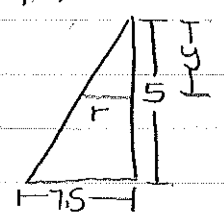
5. The base of a certain solid is the triangle with vertices at (-10, 5), (5, 5), and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.



rotate

height = 5

base = 15



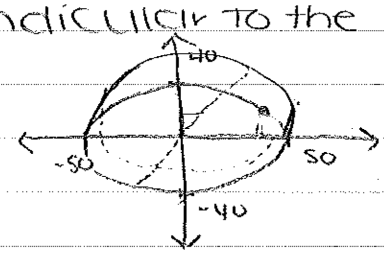
$\frac{r}{7.5} = \frac{y}{5}$

$5r = 7.5y$

$r = 1.5y \rightarrow$ side of square $= 3y$

Thus $A(y) = 9y^2$. $\int_0^5 9y^2\,dy = 3y^3\Big|_0^5 = 375$. ✓

6. As viewed from above, a swimming pool has the shape of the ellipse $\frac{x^2}{2500} + \frac{y^2}{1600} = 1$. The cross sections perpendicular to the x-axis are squares.



$x^2/a^2 + y^2/b^2 = 1$, $a = 50$, $b = 40$

$x^2 + \frac{2500y^2}{1600} = 2500$

$\frac{2500y^2}{1600} = 2500 - x^2$

$y^2 = \frac{1600}{2500}(2500 - x^2)$

$y = 4/5\sqrt{2500 - x^2}$

$A(y) = 64/25(2500 - x^2)$.

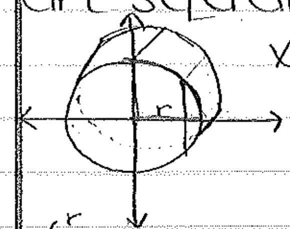
$\frac{128}{25}\int_0^{50} 2500 - x^2\,dx$

$\frac{128}{25}\left(2500x - \frac{1}{3}x^3\right)\Big|_0^{50}$

$\approx 426,666.665$

~~multiplying by 4 gives us the total area of 213,333.333~~

7. Calculate the volume of the following solid. The base is a circle of radius r centered at the origin. The cross sections perpendicular to the x-axis are squares.



$x^2+y^2=r^2$, area in terms of x

$y^2=r^2-x^2$

$y=\sqrt{r^2-x^2}$, side length $=$

$2y = 2\sqrt{r^2-x^2}$

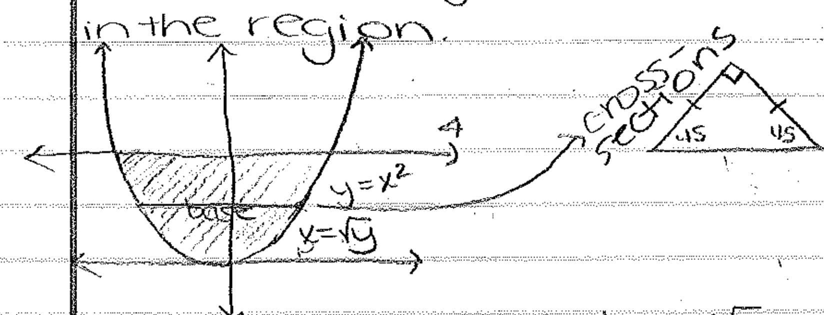
$$\int_{-r}^{r} A(x)\,dx = 2\int_0^r 4r^2-4x^2\,dx = 2\left(4r^2x-\tfrac{4}{3}x^3\right)\Big|_0^r$$

$$= 2\left(4r^3-\tfrac{4}{3}r^3\right)$$

$$= 2\left(\tfrac{8}{3}r^3\right)$$

$$= \tfrac{16}{3}r^3$$

8. Calculate the volume of the following solid. The base is the parabolic region $\{(x,y)\mid x^2\le y\le 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.



hypotenuse $=2\sqrt{y}$ $\quad s=\frac{h}{\sqrt{2}}=\frac{2\sqrt{y}}{\sqrt{2}}=\sqrt{2}\sqrt{y}=\sqrt{2y}$

$A(y)=\frac{1}{2}s^2=\frac{1}{2}(2y)=y$

$$V=\int_0^4 A(y)\,dy=\int_0^4 y\,dy=\tfrac{1}{2}y^2\Big|_0^4=\tfrac{1}{2}(16)=8.$$