Worksheet #12:
Density and Average Value & Volumes of Revolution (Disk Method)

1. Conceptual Understanding:

   (a) If the linear mass density of a rod at position x is given by the function $\rho(x)$, what integral should be evaluated to find the mass of the rod between points a and b?

   (b) If the radial mass density of a disk centered at the origin is given by the function $\rho(r)$, where r is the distance from the center point, what integral should be evaluated to find the mass of a disk of radius R?

   (c) Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the total mass of a 1-meter rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 2$.

3. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$

   (b) $f(x) = x^3,\ [-1, 1]$

   (c) $f(x) = \cos(x),\ \left[0, \dfrac{\pi}{6}\right]$

   (d) $f(x) = \dfrac{1}{x^2+1},\ [-1, 1]$

   (e) $f(x) = \dfrac{\sin(\pi/x)}{x^2},\ [1, 2]$

   (f) $f(x) = e^{-nx},\ [-1, 1]$

   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$

   (h) $f(x) = x^n$ for $n \ge 0$, $[0, 1]$

4. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1+r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5 km radius of the clearing.

5. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

6. Find the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x^5}$, $y = 0$, $x = 1$, and $x = 6$, about the x-axis.

7. Find the volume of the solid obtained by rotating the region bounded by $f(x) = \sin(x)$ and the x-axis over the interval $[0, \pi]$ about the x-axis.
   [Hint: You may use the trig identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ in order to evaluate the integral.]

8. Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $x = 0$, $y = 1$, $x = y^{11}$, about the line $y = 1$.

9. For each of the following, use disks or washers to find the integral expression for the volume of the region.

   (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.

   (b) R is the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the y-axis.

   (c) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.

   (d) Between the regions in part (a) and part (c), which volume is bigger? Why? First argue without computing the integrals, then also evaluate the integrals to check your answer.

   (e) R is the region bounded by $y = e^{-x}$, $y = 1$ and $x = 2$; about the line $y = 2$.

(f) R is the region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

10. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. $y = 0$, $y = \cos(2x)$, $x = \frac{\pi}{2}$, $x = 0$ about the line $y = -6$.

11. Find the volume of the cone obtained by rotating the region in the first quadrant under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

12. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \le x \le 1$ about the y-axis. Soda is extracted from the glass through a straw at the rate of 1/2 cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

13. The torus is the solid obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]