Worksheet 12: Math 114
Density and Average Value and Volumes of Revolution (Disk Method).

1. Conceptual Understanding:

(a) If the linear mass density of a rod at position x is given by the function $\rho(x)$, what integral should be evaluated to find the mass of the rod between points a and b?

$\int_a^b \rho(x)\,dx$

(b) If the radial mass density of a disk centered at the origin is given by the function $\rho(r)$, where r is the distance from the center point, what integral should be evaluated to find the mass of a disk of radius R?

$\int_{-R}^{R} \rho(r)\,dr$

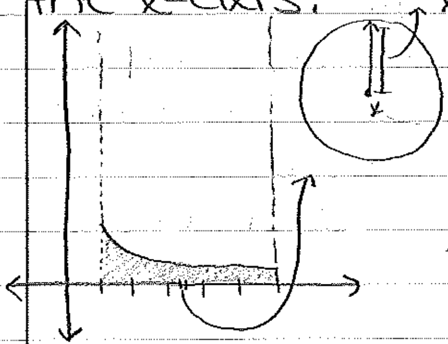
(c) write down the equation for the average value of an integrable function f(x) on [a,b].

2. Find the total mass of a 1-meter rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 2$.

$$\int_0^2 10(x+1)^{-2}\,dx = -10(x+1)^{-1}\Big|_0^2 = -10\left(\tfrac{1}{3}\right) - (-10)(1)$$

$$= -\frac{10}{3} + \frac{30}{3} = \frac{20}{3}$$

6. Find the volume obtained by rotating the region bounded by $y = 1/x^5$, $y=0$, $x=1$, and $x=6$ about the x-axis.



$1/x^5$ = radius

$A_O = \pi r^2 = \pi\left(\frac{1}{x^5}\right)^2 = \pi\left(\frac{1}{x^{10}}\right)$

$\int_1^6 \pi\left(\frac{1}{x^{10}}\right)dx$

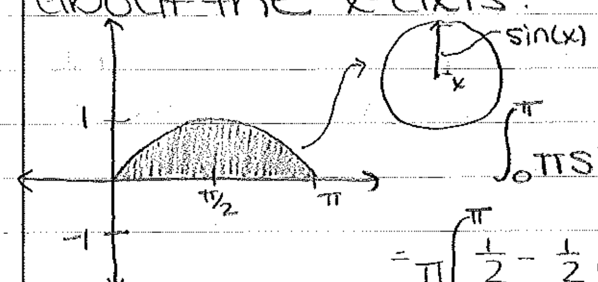
$= \pi\int_1^6 x^{-10}\,dx = \pi\left(-\frac{1}{9}x^{-9}\right)\Big|_1^6$

$= \pi\left(-\frac{1}{9}(6)^{-9} - \left(-\frac{1}{9}(1)^{-9}\right)\right)$

$= \pi(.111)$

$\approx .349$

7. Find the volume of the solid obtained by rotating the region bounded by $f(x) = \sin(x)$ and the x-axis over the interval $[0,\pi]$ about the x-axis.



$A_O = \pi r^2 = \pi\sin^2(x)$

$\int_0^\pi \pi\sin^2(x)\,dx$

$= \pi\int_0^\pi \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx$

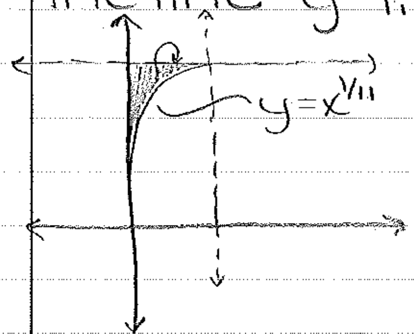
$= \pi\left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big|_0^\pi$

$= \pi\left(\frac{1}{2}\pi - \frac{1}{2}\sin(2\pi)\right)$

$= \pi\left(\frac{\pi}{2}\right)$

$= \pi^2/2 \approx 4.9348$

8. Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $x=0$, $y=1$, $x=y^{11}$, about the line $y=1$.



$A_0 = \pi r^2 = \pi x^{2/11}$

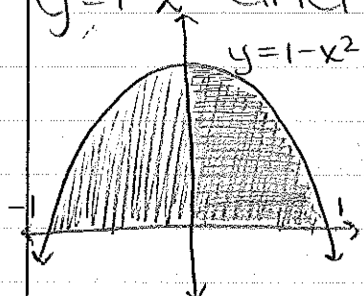
$\pi \int_0^1 x^{2/11}\,dx$

$= \pi \left(\frac{11}{13} x^{13/11}\right)\Big|_0^1$

$= 11\pi/13 \approx 2.658$

9. For each of the following, use disks or washers to find the integral expression for the volume of the region.

(a) R is the region bounded by $y=1-x^2$ and $y=0$; about the x-axis



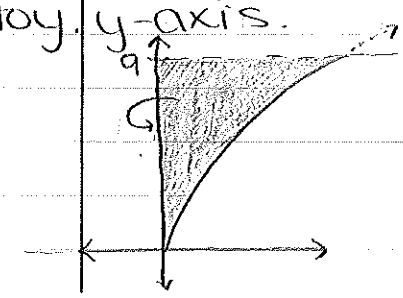
$2\pi \int_0^1 1-2x^2+x^4\,dx$

$= 2\pi \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1$

$= 2\pi\left(1 - 2/3 + 1/5\right)$

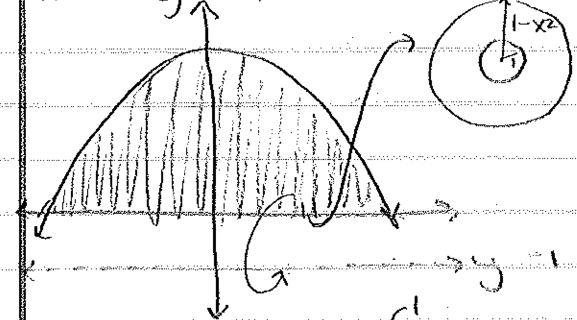
≈ 3.351

ntegrate vith espect to y. (b) R is the region bounded by $x = 2\sqrt{y}$, $x=0$, and $y=9$; about the y-axis.



$\pi \int_0^9 4y\,dy$

(c) R is the region bounded by $y=1-x^2$ and $y=0$; about the line $y=-1$

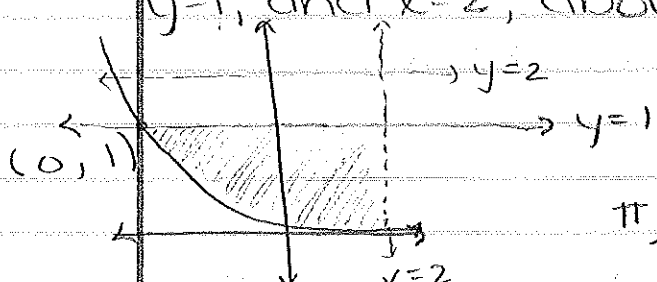


$$A=\pi(-1-1+x^2)^2 - \pi(-1-0)^2$$
$$=\pi(-2+x^2)^2-\pi$$
$$=\pi(4-4x^2+x^4-1)$$
$$=\pi(3-4x^2+x^4)$$

$$2\pi\int_0^1 3-4x^2+x^4\,dx$$
$$=2\pi\left(3x-\tfrac{4}{3}x^3+\tfrac{1}{5}x^5\right)\Big|_0^1$$
$$=2\pi(3-4/3+1/5)\approx 11.7286$$

(d) Between the regions in part (a) and part (c), which volume is bigger?
The region in part (c) has a larger volume.

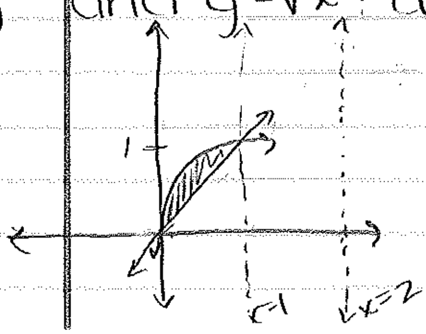
(e) R is the region bounded by $y=e^{-x}$, $y=1$, and $x=2$; about the line $y=2$.



$$A=\pi((2-e^{-x})^2-(1)^2)$$
$$\pi\int_0^1 (2-e^{-x})^2-1\,dx$$

ntegrate w.r.t y

(f) R is the region bounded by $y=x$ and $y=\sqrt{x}$; about the line $x=2$.



$x=y^2$ $A=\pi((2-y^2)^2-(2-y)^2)$

$x=y$ $$\pi\int_0^1 (2-y^2)^2-(2-y)^2\,dy$$