Math 114 : Worksheet 13

Volumes of Revolution (Shell Method)

1. Conceptual Understanding

(a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a,b]$ about the y-axis using the method of cylindrical shells.

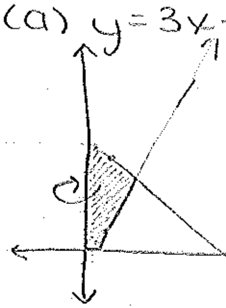
$$V = 2\pi \int_a^b x f(x)\, dx$$

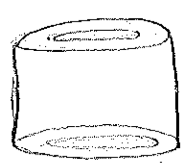
(b) If you use the disk method to compute the same volume, are you integrating with respect to x or y?

We are integrating with respect to y because the disks partition the solid with respect to y.

2. Sketch the enclosed region and use the shell method to calculate the volume of rotation about the y-axis.

(a) $y = 3x - 2$, $y = 6 - x$, $x = 0$





radius $= x$

height $= (6-x) - (3x-2)$

$= 6 - x - 3x + 2$

$= 8 - 4x$

$$V = 2\pi \int_0^2 8x - 4x^2\, dx$$

$$= 2\pi \left(4x^2 - \tfrac{4}{3}x^3\right)\Big|_0^2$$

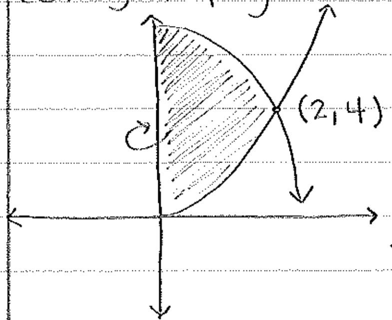
$$= 2\pi\left(4(4) - \tfrac{4}{3}(8)\right)$$

$$= 2\pi\left(\tfrac{48}{3} - 32/3\right)$$

$$= 2\pi\left(16/3\right)$$

$$= 32\pi/3$$

(b) $y = x^2$, $y = 8 - x^2$, $x = 0$ for $x \geq 0$



radius $= x$

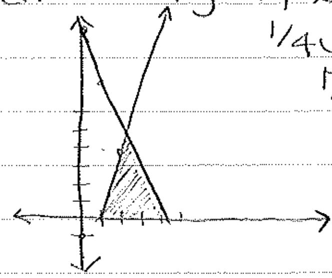
height $= (8 - x^2) - x^2$

$= 8 - 2x^2$

$$V = 2\pi \int_0^2 8x - 2x^3 \, dx$$
$$= 2\pi \left(4x^2 - \tfrac{1}{2}x^4\right)\Big|_0^2$$
$$= 2\pi \left(16 - \tfrac{1}{2}(16)\right)$$
$$= 2\pi(8)$$
$$= 16\pi$$

3. Sketch the enclosed region and use the shell method to calculate the volume of the solid when rotated about the x-axis.

(a) $x = \frac{1}{4}y + 1$, $x = 3 - \frac{1}{4}y$, $y = 0$



$\frac{1}{4}y + 1 = 3 - \frac{1}{4}y$

$\frac{1}{2}y = 2$

$y = 4$

radius $= y$

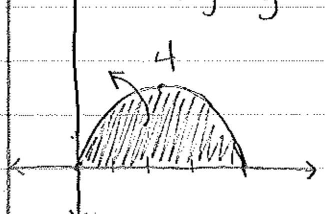
height $= (3 - \frac{1}{4}y) - (\frac{1}{4}y + 1)$

$= 2 - \frac{1}{2}y$

$$V = 2\pi \int_0^4 2y - \tfrac{1}{2}y^2 \, dy$$
$$= 2\pi \left(y^2 - \tfrac{1}{6}y^3\right)\Big|_0^4$$
$$= 2\pi \left(16 - \tfrac{1}{6}(64)\right)$$
$$= 2\pi \left(\tfrac{32}{6}\right)$$
$$= \tfrac{64\pi}{6} = \tfrac{32\pi}{3}$$

(b) $x = y(4 - y)$, $x = 0$

$= 4y - y^2$



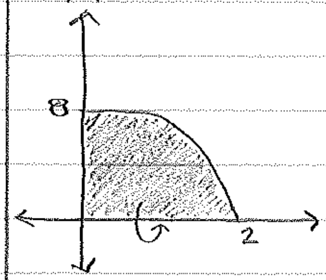
radius $= y$

height $= 4y - y^2$

$$V = 2\pi \int_0^4 4y^2 - y^3 \, dy$$
$$= 2\pi \left(\tfrac{4}{3}y^3 - \tfrac{1}{4}y^4\right)\Big|_0^4$$
$$= 2\pi \left(\tfrac{4}{3}(64) - 64\right) = \tfrac{128}{3}\pi$$

* 4. Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of $f(x) = 8 - x^3$ for $0 \le x \le 2$ about:

(a) the x-axis (Disk Method)



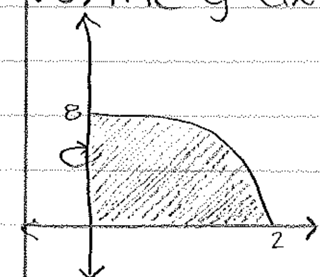
radius $= 8 - x^3$

$$V = \pi \int_0^2 (8-x^3)^2\,dx$$
$$= \pi \int_0^2 64 - 16x^3 + x^6\,dx$$
$$= \pi \left(64x - 4x^4 + \tfrac{1}{7}x^7\right)\Big|_0^2$$
$$= \pi (128 - 64 + 128/7)$$
$$= \pi (64 + 128/7)$$

$$V = 2\pi \int_0^8 y\sqrt[3]{8-y}\,dy$$

integrate by substitution

(b) the y-axis



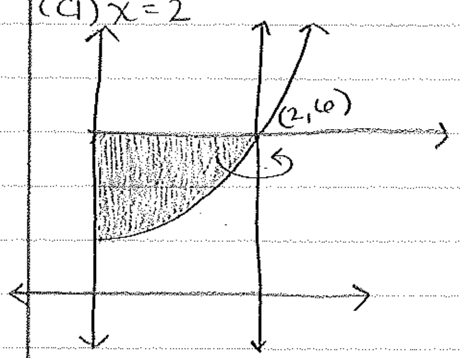
$$V = 2\pi \int_0^2 x(8-x^3)\,dx$$
$$= 2\pi \left(4x^2 - \tfrac{1}{5}x^5\right)\Big|_0^2$$
$$= 2\pi (16 - 32/5)$$

$$V = \pi \int_0^8 \left(\sqrt[3]{8-y}\right)^2 dy$$

integrate by substitution

5. Use the Shell Method to find the volume obtained by rotating the region bounded by $y = x^2 + 2$, $y = 6$, $x = 0$, and $x = 2$, about the following axes:

(a) $x = 2$



radius: $(2 - x)$

height: $(6 - x^2 - 2)$

$$V = 2\pi \int_0^2 (2-x)(4-x^2)\,dx$$
$$= 2\pi \int_0^2 8 - 4x - 2x^2 + x^3\,dx$$
$$= 2\pi \left(8x - 2x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4\right)\Big|_0^2$$
$$= 2\pi \left(16 - 8 - \tfrac{16}{3} + 4\right)$$
$$= 2\pi \left(12 - 16/3\right)$$

(b) $x=-3$ radius $=(x-(-3))=(x+3)$

height $=(6-x^2-2)$

$=(4-x^2)$

$V = 2\pi\int_0^2 (x+3)(4-x^2)\,dx$

$= 2\pi\int_0^2 12+4x-3x^2-x^3\,dx$

$= 2\pi\left(12x+2x^2-x^3-\frac{1}{4}x^4\right)\Big|_0^2$

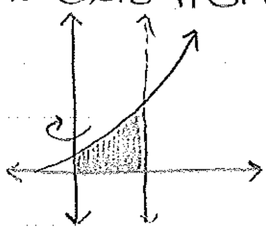
$= 2\pi(24+8-8-4)$

$= 2\pi(20)$

$= 40\pi$

6. Find the volume of the solid obtained by rotating the region about the y-axis.

(a) The region bounded by $f(x)=e^x$ and the x-axis from $0\le x\le 2$.



$V = 2\pi\int_0^2 x(e^x)\,dx$ $u=x$ $dv=e^x dx$

$= 2\pi\left[xe^x-\int_0^2 e^x dx\right]$ $du=dx$ $v=e^x$

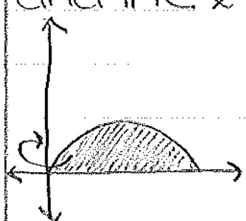
$= 2\pi(2e^2-e^x)\Big|_0^2$

$= 2\pi(2e^2-e^2+1)$ (correct).

$=(4\pi-1)e^2 \approx 85.46$

$+1$

(b) The region bounded by $f(x)=\sin(x)$ and the x-axis from $0\le x\le \pi$.



$V = 2\pi\int_0^\pi x\sin(x)\,dx$ $u=x$ $dv=\sin(x)dx$

$du=dx$ $v=-\cos(x)$

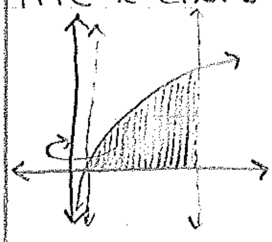
$= 2\pi\left(-x\cos(x)+\int_0^\pi \cos(x)\,dx\right)$

$= 2\pi(-x\cos(x)+\sin(x))\Big|_0^\pi$

$= 2\pi(-\pi(-1)+0)$

$= 2\pi^2$

*(c) The region bounded $f(x) = \ln(x)$ and the x-axis from $1 \leq x \leq 3$.



$$V = 2\pi \int_1^3 x \ln(x)\, dx$$

$$= 2\pi \left(\frac{x^2 \ln(x)}{2} - \frac{x^2}{4} \right) \Big|_1^3$$

$$= 2\pi \left(\frac{9\ln(3)}{2} - \frac{9}{4} + \frac{1}{4} \right)$$

$$= 2\pi \left(\frac{9\ln(3)}{2} - 2 \right)$$

$$\approx 18.496$$