Math 114: Worksheet 14

Work and Trigonometric Integrals

1. Conceptual Understanding:

(a) Describe and define work. What are its units? What is the difference between work and force?

Work is the amount of force necessary to move an object a certain distance. Force is what causes an object to move (e.g. push or pull). Work is usually measured in joules.

(b) Determine the work done in lifting a 1 kg weight through a distance of 1 meter near the surface of the earth, maintaining a constant velocity.

$W = F(d) = (mg)(d)$

$= (1\,kg)(9.8\,m/s^2)(1\,m)$

$= 9.8\,J$

(c) How much work is done in lifting a 1kg weight up 1m at a constant velocity and then lowering it back 1m at a constant velocity?

$W = 0 = 9.8\,J - 9.8\,J$, 0 work total.

2. Determine the work done in lifting a 500 kg elevator 1000 m to the top floor of a building. How much work is done lowering a 500 kg elevator 1000 m from the top floor of the building to the ground floor? How much work is done making the

round trip?

$W = (mg)d$

$= 500\,kg(9.8\,m/s^2)(1000\,m)$

$= 4{,}900{,}000\ J$ (lifting)

$W = 500\,kg(-9.8\,m/s^2)(1000\,m)$

$= -4{,}900{,}000\ J$ (lowering)

net work $= 0$

3. A force of 50 N will stretch a spring from its natural length of 5cm to 15cm. How much work will be done in stretching the spring from 15cm to 30cm?

Hooke's Law: $F(x) = kx$, (the force required to stretch a spring a distance of x meters from its natural length).

$50 = k(.10) \rightarrow k = 500$

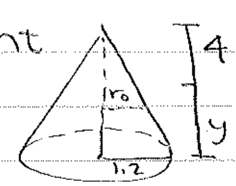
$W = \int_{.15}^{.3} 500x\,dx$

$= 250x^2 \Big|_{.15}^{.3}$

$= 22.5 - 5.625$

$= 16.875\ J$

4. Calculate the work against gravity required to build a right circular cone of height 4m and base radius 1.2 meters out of a lightweight material of density 600 kg/m³.



work of lifting layer to height y

$\approx 5880\pi(1.2-.24y)dy$

$\frac{1.2}{4} = \frac{r_0}{4-y}$

$6 - 1.2y = 5r$ volume of layer = area × width

$1.2 - .24y = r$ $= \pi((.3)(4-y))^2 dy$

mass of layer = density × volume

$= 600\pi((.3)(4-y))^2 dy$

force on layer = g × mass

$= 5880\pi((.3)(4-y))^2 dy$

$W = 529.2\pi \int_0^4 (16 - 8y + y^2) y \, dy$

$= 11,289.6\pi$

5. Consider a rectangle tank of water that is 5 m tall and has a base of size 8×4 meters. It has a spout on its top surface. Calculate the work required to pump all of the water out of the tank. Density of the water is 1000 kg/m³.

force on layer = g × density × A(y) dy

$\approx 9.8(1000)(32)\,dy$

work on layer $= 9800(32)dy\,(5-y)$

$= 1,568,000 - 313600\,y\,dy$

$W = \int_0^5 1568{,}000 - 313600\,y\,dy$

$= 1,568,000\,y - 156,800\,y^2 \Big|_0^5$

$= 7,840,000 - 3,920,000$

$= 3,920,000$ J

6. Evaluate the following integrals.

(a) $\int \cos^2(x)\,dx$

$= \int \frac{1}{2}(1 + \cos 2x)\,dx$

$= \int \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

(b) $\int_0^{\pi/2} \sin^2(x)\cos^2(x)\,dx$

$= \int_0^{\pi/2} (1-\cos^2(x))\cos^2(x)\,dx$

$= \int_0^{\pi/2} \cos^2(x) - \cos^4(x)\,dx$

$= \int_0^{\pi/2} \frac{1}{2}+\frac{1}{2}\cos(2x) - (\frac{1}{2}+\frac{1}{2}\cos(2x))^2\,dx$

$= \int_0^{\pi/2} \frac{1}{2}+\frac{1}{2}\cos(2x) - (\frac{1}{4}+\frac{1}{2}\cos(2x)+1/4\cos^2(2x))\,dx$

$= \int_0^{\pi/2} \frac{1}{4} - 1/4\cos^2(2x)\,dx$

$= \int_0^{\pi/2} \frac{1}{4} - 1/4(\frac{1}{2}+\frac{1}{2}\cos(4x))\,dx$

$= \int_0^{\pi/2} 1/8 - 1/8\cos(4x)\,dx$

$= (1/8x - \frac{1}{32}\sin(4x))\Big|_0^{\pi/2}$

$= \frac{\pi}{16} - \frac{1}{32}\underbrace{\sin(2\pi)}_{0}$

$= \pi/16.$

(c) $\int \sin^3(x)\cos^2(x)\,dx$

$= \int \sin^2(x)\sin(x)\cos^2(x)\,dx$

$= \int (1-\cos^2(x))\sin(x)\cos^2(x)\,dx$

$= \int (\cos^2(x)-\cos^4(x))\sin(x)\,dx$ let $u=\cos x$

$= -\int u^2-u^4\,du$ $du=-\sin x\,dx$

$= -(\frac{1}{3}u^3-\frac{1}{5}u^5)$

$= 1/5u^5 - 1/3u^3$

$= 1/5\cos^5(x) - 1/3\cos^3(x) + C$

(d) $\int x^2\cos(x)\,dx$ let $u=x^2$ $dv=\cos(x)dx$

$du=2x\,dx$ $v=\sin(x)$

$= x^2\sin(x) - \int 2x\sin(x)\,du$ $u'=2x$ $dv'=\sin(x)dx$

$du'=2$ $v'=-\cos(x)$

$-(-2x\cos(x)+\int 2\cos(x)dx)$

$= x^2\sin(x)+2x\cos(x)-2\sin(x)+C$

(e) $\int e^x\cos x\,dx$ $u=e^x$, $du=e^x dx$ $dv=\cos x\,dx$ $v=\sin x$

$= e^x\sin x - \int \sin x\, e^x\,dx$ $u=e^x$ $du=e^x dx$ $dv=\sin x\,dx$

$= e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$ $v=-\cos x\,dx$

$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x$

$\int e^x\cos x\,dx = \frac{1}{2}(e^x\sin x + e^x\cos x) + C$