Math 114: Worksheet 15

Trigonometric Integrals

1. Evaluate the following integrals.

(a) $\int \tan^2(x)\,dx$

$= \int \sec^2(x) - 1\,dx$

$= \int \sec^2(x)\,dx - \int 1\,dx$

$= \tan(x) - x + C$

(b) $\int \frac{\sin(x)}{\cos^3(x)}\,dx$

$= \int \tan(x)\sec^2(x)\,dx$ let $u = \tan x$

$= \int u\,du$ $du = \sec^2 x\,dx$

$= \frac{1}{2}u^2 + C$

$= \frac{1}{2}\tan^2 x + C$

(c) $\int \tan^5(x)\sec^3(x)\,dx$

$= \int (\sec^2 x - 1)^2(\sec^2(x))\sec x \tan x\,dx$

$= \int (u^2-1)^2(u^2)\,du$ let $u = \sec x$

$= \int (u^4 - 2u^2 + 1)u^2\,du$ $du = \sec x \tan x\,dx$

$= \int u^6 - 2u^4 + u^2\,du$

$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$

$= \frac{1}{7}\sec^7(x) - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C$

(d) $\int \sin(8x)\cos(5x)\,dx$ $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta)$

$= \int \frac{1}{2}\sin(13x) + \frac{1}{2}\sin(3x)\,dx$

$= \frac{1}{2}\int \sin(13x)\,dx + \frac{1}{2}\int \sin(3x)\,dx$

$= \frac{1}{2}(1/13) - \cos(13x) + 1/2(1/3)(-\cos(3x)) + C$

$= -1/26 \cos(13x) - 1/6 \cos(3x) + C.$

(e) $\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx$

$= \int {-\sec^2 x}/\sec^2 x\,dx$

$= \int -1\,dx$

$= -x + C$

(f) $\int x\sec^2(x^2)\tan^4(x^2)\,dx$ let $u=x^2$ $du=2x\,dx$

$= \frac{1}{2}\int \sec^2(u)\tan^4(u)\,du$ let $v=\tan(u)$ $dv=\sec^2(u)\,du$

$= \frac{1}{2}\int v^4\,dv$

$= \frac{1}{2}(\frac{1}{5}v^5 + c)$

$= \frac{1}{10}\tan^5(u) + c$

$= \frac{1}{10}\tan^5(x^2) + c$

(g) $\int_{-\pi/4}^{\pi/4} \tan^3(x)\,dx$

$= \int_{-\pi/4}^{\pi/4} \tan^2(x)\tan(x)\,dx$

$= \int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan(x)\,dx$

$= \int_{-\pi/4}^{\pi/4} \sec^2 x\tan x\,dx - \int_{-\pi/4}^{\pi/4} \frac{\sin(x)}{\cos(x)}\,dx$ let $u=\tan x$

$= \int_{-\pi/4}^{\pi/4} u\,du + \int_{-\pi/4}^{\pi/4} \frac{1}{v}\,dv$ $du=\sec^2 x\,dx$

$= \frac{1}{2}u^2 \Big|_{-\pi/4}^{\pi/4} + \ln|v|\Big|_{-\pi/4}^{\pi/4}$ let $v=\cos(x)$

$dv=-\sin(x)\,dx$

$= \frac{1}{2}\tan^2(x)\Big|_{-\pi/4}^{\pi/4} + \ln|\cos(x)|\Big|_{-\pi/4}^{\pi/4}$

$= \frac{1}{2}((1)^2-(-1)^2) + \ln|\frac{\sqrt{2}}{2}| - \ln|-\frac{\sqrt{2}}{2}|$

$= \frac{1}{2}(1-1) + \ln|1|$

$= 0 + 0$

$= 0$

(h) $\int_{\pi/4}^{\pi/2} \cot^3(x)\,dx$

$= \int_{\pi/4}^{\pi/2} \cot^2(x)\cot(x)\,dx$

$= \int_{\pi/4}^{\pi/2} (\csc^2(x)-1)\cot(x)\,dx$

$= \int_{\pi/4}^{\pi/2} \csc^2(x)\cot(x)\,dx - \int_{\pi/4}^{\pi/2} \cot(x)\,dx$ ① u-sub w/ $u=\cot(x)$

$= -\frac{1}{2}\cot^2(x)\Big|_{\pi/4}^{\pi/2} - \ln|\sin x|\Big|_{\pi/4}^{\pi/2}$

$= -\frac{1}{2}(0) - (-\frac{1}{2}) - \ln|1| + \ln|\sqrt{2}/2|$

$= 1/2 + \ln|\sqrt{2}/2|$

2. Prove that for any two integers m and n $\frac{1}{\pi}\int_0^{2\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 0 & m\neq n \\ 1 & m=n \end{cases}$

$\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m-n)x)+\cos((m+n)x)]$

$\frac{1}{\pi}\int_0^{2\pi}\cos(mx)\cos(nx)\,dx$

$=\frac{1}{2\pi}\int_0^{2\pi}\cos((m-n)x)+\cos((m+n)x)\,dx$

IF $m=n$:

$\frac{1}{2\pi}\int_0^{2\pi} 1+\cos(m+n)x\,dx$

$=\frac{1}{2\pi}\left(x+\frac{1}{2m}\sin(2mx)\right)\Big|_0^{2\pi}$

$=\frac{1}{2\pi}\left[\left(2\pi+\frac{1}{2m}\sin(4m\pi)\right)-\left(0+\frac{1}{2m}(0)\right)\right]$

$=\frac{1}{2\pi}[2\pi+0-0+0]$

$=1$

IF $m\neq n$

$=\frac{1}{2\pi}\int_0^{2\pi}\cos((m-n)x)+\cos((m+n)x)\,dx$

$=\frac{1}{2\pi}\left[\frac{1}{m-n}\sin((m-n)x)+\frac{1}{(m+n)}\sin((m+n)x)\right]\Big|_0^{2\pi}$

$=\frac{1}{2\pi}\left[\frac{1}{m-n}(0)+\frac{1}{m+n}(0)-\frac{1}{m-n}(0)-\frac{1}{m+n}(0)\right]$

$=0$ ✓.