Math 114: Worksheet #16

Review for Exam 2

1. Power, Maclaurin, and Taylor Series.

(a) Find the Maclaurin series for $x^2/1+x$.

$\frac{1}{1+x} = \frac{1}{1-(-x)} = x^2 \sum_{n\geq 0} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$

(b) Find the Taylor series for cos x about $a = \pi/2$.

$f(x) = \cos(x) \quad f(\pi/2) = 0$

$f'(x) = -\sin(x) \quad f'(x) = -1$

$f''(x) = -\cos(x) \quad f''(x) = 0$

$f'''(x) = \sin(x) \quad f'''(x) = 1$

$(x - \pi/2) + \frac{(x - \pi/2)^3}{3!} - \frac{(x - \pi/2)^5}{5!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}$

(c) Find the Taylor series centered at $c = 0$ of $2/4-3x$ and determine its radius of convergence.

$2\left(\frac{1}{4-3x}\right) = \frac{1}{2}\left(\frac{1}{1-\frac{3}{4}x}\right) = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n x^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{3}{4}x\right)^n$

$\lim_{n\to\infty} \left| \frac{\frac{1}{2}\left(\frac{3}{4}x\right)^{n+1}}{\frac{1}{2}\left(\frac{3}{4}x\right)^n} \right| = \lim_{n\to\infty} \left| \frac{3}{4}x \right| \overset{?}{<} 1$, when $|x| < 4/3$

with radius of convergence $4/3$?

(d) Find the Taylor series centered at 0 of the function $f(x) = \ln(x+5)$.

Note $f'(x) = 1/x+5 = \frac{1}{5}\left(\frac{1}{1+x/5}\right)$

$\int \frac{1}{5} \sum_{n=0}^{\infty} (-1)^n (x/5)^n \, dx = \frac{1}{5} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{5}\right)^n \frac{x^{n+1}}{n+1}$

$= \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{1}{5}\right)^n \; x^n \; /n$

(e) Find the Taylor series centered at zero of the function $g(x) = x^3 \ln(x^2+5)$

2. $T_3(x) = 1 - \frac{1}{\pi^2}(x)^2 \cdot \frac{1}{2!}$

$f(x) = \cos(x/\pi)$ $\quad f(0) = \cos(0) = 1$

$f'(x) = \frac{-1}{\pi}\sin(x/\pi)$ $\quad f(0) = \sin(0) = 0$

$f''(x) = \frac{-1}{\pi^2}\cos(x/\pi)$ $\quad f''(0) = -1/\pi^2$

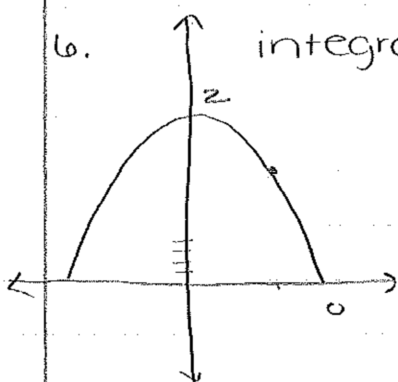
$f'''(x) = 1/\pi^3 \sin(x/\pi)$ $\quad f'''(0) = 0$

$f''''(x) = 1/\pi^4 \cos(x/\pi)$ $\quad f''''(0) = 1/\pi^4$

3. $T_n(x) = \sum_{k=0}^{n} \left(\frac{(3x)^n}{n!}\right)$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$

6. integrate w.r.t y



$x = \sqrt{2-y}$

$\int_0^2 (2\sqrt{2-y})^2\, dy$

$= \int_0^2 4(2-y)\, dy$

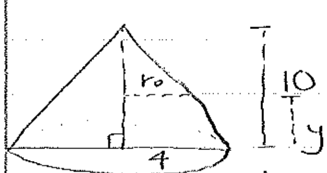
$= \int_0^2 8 - 4y\, dy$

$= 8y - 2y^2 \Big|_0^2$

$= 16 - 2(4)$

$= 8$

7. (a)



$\frac{10-y}{r} = \frac{10}{4}$

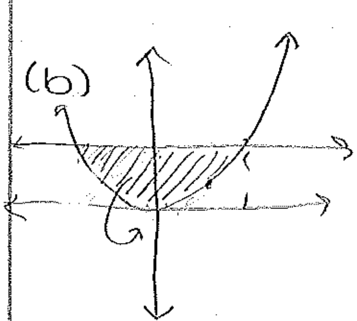
$4 - \frac{2}{5}y = r$

$A_0 = \pi(4 - 2/5\, y)^2$

$\pi \int_0^{10} 16 - \frac{16}{5}y + \frac{4}{25}y^2\, dy = \pi\left(16y - \frac{8}{5}y^2 + \frac{4}{75}y^3\right)\Big|_0^{10}$

$\pi\left(160 - 160 + \frac{160}{3}\right) = \frac{160}{3}\pi.$

(b)



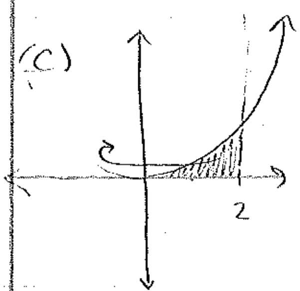
$$\pi\int_{-1}^{1}(1)^2-(x^2)^2\,dx$$

$$\pi\int_{-1}^{1}1-x^4\,dx = \pi\left(x-\tfrac{1}{5}x^5\right)\Big|_{-1}^{1}$$

$$= 2\pi\left(x-\tfrac{1}{5}x^5\right)\Big|_0^1 = 2\pi(1-1/5)$$

$$= 8\pi/5$$

(c)



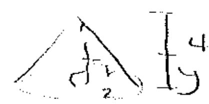
$$2\pi\int_0^2(\underbrace{3x^2}_{\text{height}})\underbrace{x}_{\text{radius}}\,dx$$

$$= 2\pi\int_0^2 3x^3\,dx$$

$$= 2\pi\left(\tfrac{3}{4}x^4\right)\Big|_0^2$$

$$= 2\pi(12) = 24\pi$$

8. Work.



work lifting layer to height y:

volume of layer: $\pi(2-y/2)^2\,\Delta y$

mass: $600\pi\,(2-y/2)^2\,dy$

force $= 5880\pi(2-y/2)^2\,dy$

work $= 5880\pi\left(4y-2y^2+\frac{y^3}{4}\right)$

$$\frac{4-y}{r}=\frac{4}{2}$$

$$4-y=2r$$

$$r=2-y/2$$

$$5880\pi\int_0^4\left(4y-2y^2+y^3/4\right)dy$$

$$= 5880\pi\left(2y^2-\tfrac{2}{3}y^3+\tfrac{1}{16}y^4\right)\Big|_0^4$$

$$= 5880\pi\left(32-\tfrac{2}{3}(64)+16\right) = 5880\pi\left(48-\tfrac{128}{3}\right) = 31360\pi$$

9. (a) $\int \sin^2(x)\cos^3(x)\,dx$

$= \int \sin^2(x)\cos^2(x)\cos x\,dx$

$= \int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$

$= \int (\sin^2(x)-\sin^4(x))\cos(x)\,dx$ let $u = \sin x$, $du = \cos x\,dx$

$= \int u^2 - u^4\,du$

$= 1/3\,u^3 - 1/5\,u^4$

$= 1/3 \sin^3 x - 1/5 \sin^5 x + C.$

(b) $\int \tan^3 x \sec^3 x\,dx$

$= \int \tan^2 x \sec^2 x\,(\tan x \sec x)\,dx$

$= \int (\sec^2 x - 1)\sec^2 x\,(\tan x \sec x)\,dx$

$= \int \sec^4 x - \sec^2 x\,(\tan x \sec x)\,dx$ let $u = \sec x$

$= \int u^4 - u^2\,du$ $du = \tan x \sec x\,dx$

$= 1/5\,u^5 - 1/3\,u^3$

$= 1/5 \sec^5 x - 1/3 \sec^3 x + C$