# Math 114: Worksheet 17

## Integration by Trig Substitution

### 1. Conceptual Understanding

(a) Given that $\sin^2\phi + \cos^2\phi = 1$, prove that $\tan^2\phi + 1 = \sec^2\phi$

consider $\sin^2\phi + \cos^2\phi = 1$, now divide both sides by $\cos^2\phi$.

$$\frac{\sin^2\phi + \cos^2\phi}{\cos^2\phi} = \frac{1}{\cos^2\phi}$$

$$\tan^2\phi + 1 = \sec^2\phi. \qquad \square$$

(b) Given $x = a\sin\phi$ with $a > 0$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, show that $\sqrt{a^2 - x^2} = a\cos\phi$.

$\sin^2\phi + \cos^2\phi = 1$

$a^2\sin^2\phi + a^2\cos^2\phi = a^2$, let $x = a\sin\phi$

$x^2 + a^2\cos^2\phi = a^2$

$a^2\cos^2\phi = a^2 - x^2$

$a\cos\phi = \sqrt{a^2 - x^2}$

(c) Given $x = a\tan\phi$ with $a > 0$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, show that $\sqrt{a^2 + x^2} = a\sec\phi$

$a^2\tan^2\phi + a^2 = a^2\sec^2\phi$, let $x = a\tan\phi$

$x^2 + a^2 = a^2\sec^2\phi$

$\sqrt{a^2 + x^2} = a\sec\phi$

(d) Given $x = a\sec\phi$ with $a > 0$ and $0 \le \phi < \pi/2$ or $\pi \le \phi < 3\pi/2$, show that $\sqrt{x^2 - a^2} = a\tan\phi$

$a^2\tan^2\phi + a^2 = a^2\sec^2\phi$, let $x = a\sec\phi$

$a^2\tan^2\phi + a^2 = x^2$

$a^2\tan^2\phi = x^2 - a^2$

$a\tan\phi = \sqrt{x^2 - a^2}$

2. Compute the following integrals.

(a) $\int_0^{4(\sin\phi)^3} / \sqrt{16-u^2}\, du$ Let $u = 4\sin\phi$

$du = 4\cos\phi\, d\phi$

$\int_0^2 \frac{64\sin^3\phi}{4\cos\phi}(4\cos\phi)\,d\phi$ $4\cos\phi = \sqrt{16-u^2}$

$= \int_0^2 64\sin^3\phi\, d\phi = -\int_0^2 64(1-u^2)\,du$ $u = \cos\phi$

$du = -\sin\phi\, d\phi$

$= -64\int_0^2 1-u^2\, du = -64\,(u - \frac{1}{3}u^3)\Big|_0^2$

$= \frac{1}{3}(4\cos\phi)^3 - 16(4\cos\phi)\Big|_0^2 = \frac{1}{3}\sqrt{16-u^2}^{\,3} - 16(\sqrt{16-u^2})\Big|_0^2$

$= \left(\frac{1}{3}(\sqrt{12})^3 - 16(\sqrt{12})\right) - \left(\frac{1}{3}(4)^3 - 16(4)\right)$

$= \frac{1}{3}(2\sqrt{3})^3 - 32\sqrt{3} - 64/3 + 64$

$= 8/3\sqrt{3} - 32\sqrt{3} - 64/3 + 64 \approx \cancel{28.8103}\ 1.097.$

(b) $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$ Let $x = 5\sin\phi$

$dx = 5\cos\phi\, d\phi$

$5\cos\phi = \sqrt{25-x^2}$

$= \int \frac{1}{25\sin^2\phi(5\cos\phi)}(5\cos\phi\, d\phi)$

$= \frac{1}{25}\int \csc^2\phi\, d\phi = -\frac{1}{25}\cot\phi + C = -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$

(c) $\int \frac{x^3}{\sqrt{64+x^2}}\,dx$ $x = 8\tan\phi$, $dx = 8\sec^2\phi\, d\phi$

$8\sec\phi = \sqrt{64+x^2}$

$= \int \frac{8^3\tan^3\phi}{8\sec\phi}\, 8\sec^2\phi\, d\phi$

$= \int 512\tan^2\phi\tan\phi\sec\phi\, d\phi$ Let $u = \sec\phi$

$= 512\int(\sec^2\phi - 1)\tan\phi\sec\phi\, d\phi$ $du = \sec\phi\tan\phi\, d\phi$

$= 512\int u^2 - 1\, du$

$= \frac{512}{3}u^3 - 512u + C = \frac{512}{3}\sec^3\phi - 512\sec\phi + C$

$= \frac{1}{3}(8\sec\phi)^3 - 64(8\sec\phi) + C$

$= \frac{1}{3}(\sqrt{64+x^2})^3 - 64(\sqrt{64+x^2}) + C$

(d) $\int_0^1 \sqrt{x^2+1}\, dx$ let $x = \tan(\phi)$, $dx = \sec^2(\phi)dx$

$\sec\phi = \sqrt{x^2+1}$

$= \int_0^1 \sqrt{\tan^2+1}\, \sec^2\phi\, d\phi$

$= \int_0^1 \sec\phi \sec^2\phi d\phi$

$= \int_0^1 \sec^3\phi\, d\phi$ (integration by parts

note: $\int \sec x = \ln|\sec x + \tan x|$

$\int_0^1 \sec^3\phi = \sec\phi\tan\phi - \int_0^1 \sec^3\phi + \int_0^1 \sec\phi\, d\phi$

$2\int_0^1 \sec^3\phi = \sec\phi\tan\phi + \ln|\sec\phi + \tan\phi|$

$\int_0^1 \sec^3\phi = \frac{1}{2}\sec\phi\tan\phi + \ln|\sec\phi + \tan\phi|\Big|_0^1$

$\frac{1}{2}(\sqrt{x^2+1})x + \ln(\sqrt{x^2+1}+x)\Big|_0^1$

$(\frac{1}{2}(\sqrt{2}) + \ln(\sqrt{2}+1)) - \ln(1) \to 0$

$\sqrt{2}/2 + \ln(\sqrt{2}+1)$.

(e) $\int \frac{x}{\sqrt{x^2+1}}\, dx$ let $x = \tan\phi$, $dx = \sec^2\phi\, d\phi$

$\sec\phi = \sqrt{x^2+1}$

$= \int \frac{\tan\phi}{\sec\phi} \sec^2\phi\, d\phi$

$= \int \tan\phi \sec\phi\, d\phi$

$= \sec\phi + C$

$= \sqrt{x^2+1} + C$.

3. Let $a, b > 0$. Prove that the area enclosed by the ellipse $x^2/a + y^2/b = 1$ is $\pi ab$.

$y^2 = b^2(1 - x^2/a^2) = (\frac{b}{a})^2(a^2 - x^2)$

$y = \pm(b/a)\sqrt{a^2 - x^2}$ (Take the top branch and integrate from $x=0$ to $x=a$. Multiply that by 4 to get the whole area.

$4(b/a)\int_0^a \sqrt{a^2-x^2}\,dx \qquad x=a\sin(\phi) \quad \phi=\arcsin(\tfrac{x}{a})$

$t(0)=\arcsin(0)=0$

$t(a)=\arcsin(1)=\pi/2$

$dx=a\cos(\phi)\,d\phi$

$=4(\tfrac{b}{a})\int_0^{\pi/2} a\cos(\phi)(a\cos\phi)\,d\phi$

$=4(\tfrac{b}{a})\int_0^{\pi/2} a^2\cos^2\phi)\,d\phi$

$=4ab\int_0^{\pi/2}\cos^2\phi\,d\phi$

$=4ab\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\phi)\,d\phi$

$=2ab(\phi+\tfrac{1}{2}\sin 2\phi\Big|_0^{\pi/2}$

$=2ab((\pi/2+0)-(0+0))$

$=2ab(\pi/2)$

$=\pi ab.$