Math 114: Worksheet #20

Arc Length and Surface Area

1. Conceptual Understanding:

(a) Write down the formula for the arclength of a function f(x) over the interval [a,b] including the required conditions on f(x).

Assume that $f'(x)$ exists and is continuous on [a,b]. Then the arc length S, of $y=f(x)$ over [a,b] is equal to $S=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$.

(b) Write down the formula for the surface area of the surface obtained by rotating the graph of f(x) about the x-axis for $a \leq x \leq b$. How would this formula change if the graph were instead rotated about $y=c$?

Assume that $f(x) \geq 0$ and that $f'(x)$ exists and is continuous on [a,b] the surface area S of the surface obtained by rotating the graph of f(x) about the x-axis for $a \leq x \leq b$ is equal to $S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$.

Rotating about $y=c$.

$S = 2\pi \int_a^b (c-f(x))\sqrt{1+[f'(x)]^2}\,dx$

2. Find an integral expression for the arc length of the following curves. Do not evaluate the integrals.

(a) $f(x)=\sin(x)$ from $x=0$ to $x=2$.

$S=\int_0^2 \sqrt{1+[\cos^2(x)]}\,dx$

(b) $f(x) = x^4$ from $x=2$ to $x=6$.

$S = \int_2^6 \sqrt{1+(4x^3)^2}\,dx$

(c) $x^2+y^2=1$

$y^2 = 1-x^2$

$y = \sqrt{1-x^2}$

$f(x) = \sqrt{1-x^2}$

$f'(x) = \frac{1}{2}(1-x^2)^{-1/2}(2x)$

$= x(1-x^2)^{-1/2}$

$S = \int \sqrt{1+(x(1-x^2)^{-1/2})^2}\,dx$

bounds?

$S = 4\int_0^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$

3. Find the arclength of the following curves.

(a) $f(x) = x^{3/2}$ from $x=0$ to $x=2$.

$S = \int_0^2 \sqrt{1+(3/2\,x^{1/2})^2}\,dx$

$= \int_0^2 \sqrt{1+\frac{9}{4}x}\,dy$

$= \int_0^2 (1+9/4\,x)^{1/2}\,dx$

$= \frac{2}{3}(1+9/4\,x)^{3/2}(4/9)\Big|_0^2$

$= \frac{8}{27}(1+9/4\,x)^{3/2}\Big|_0^2$

$= 8/27(1+9/4(2))^{3/2} - 8/27\,(1)^{3/2}$

$= 8/27(\,11/2\,)^{3/2} - 8/27 \approx 3.52$

(b) $f(x) = \ln(\cos(x))$ from $x=0$ to $x=\pi/3$.

$f'(x)^2 = \left(\frac{1}{\cos(x)}\cdot(-\sin(x))\right)^2 = \sin^2(x)/\cos^2(x)$.

$S = \int_0^{\pi/3} \sqrt{\frac{\cos^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2 x}}\,dx = \int_0^{\pi/3} \sqrt{1/\cos^2 x}\,dx$

$= \int_0^{\pi/3} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/3}$

$= \ln|2+\sqrt{3}| - \ln|1| = \ln|2+\sqrt{3}|$

4. Set up a function $s(t)$ that gives the arclength of the curve $f(x)=2x+1$ from $x=0$ to $x=t$. Find $s(4)$.

$f'(x)=2$ $\qquad s(t)=\int_0^t \sqrt{1+4}\,dx$

$f'(x)^2=4$

$$=\int_0^t \sqrt{5}\,dx$$

$$s(4)=\int_0^4 \sqrt{5}\,dx$$

$$=\sqrt{5}x\Big|_0^4$$

$$=4\sqrt{5}$$

5. Calculate the arc length of $f(x)=x^2$ over $[0,1]$.

$f'(x)=2x$ $\qquad s=\int_0^1 \sqrt{1+4x^2}\,dx$ $\qquad$ let $x=\frac{1}{2}\tan\phi$

$(f'(x))^2=4x^2$ $\qquad\qquad dx=\frac{1}{2}\sec^2\phi\,d\phi$

$$s=\int_0^1 \tfrac{1}{2}\sqrt{1+\tan^2\phi}\,\sec^2\phi\,d\phi$$

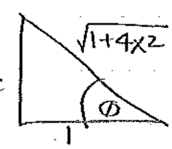
$$s=\tfrac{1}{2}\int_0^1 \sec\phi\sec^2\phi\,d\phi$$

$$s=\tfrac{1}{2}\int_0^1 \sec^3\phi\,d\phi$$

$$=1/4\tan\phi\sec\phi+1/4\ln|\sec\phi+\tan\phi|\Big|_0^1$$

Substitution $\quad=\frac{1}{2}\tan\phi\,\frac{1}{2}\sec\phi$

yields $\quad=1/2\,x\sqrt{1+4x^2}+1/4\ln|2x+\sqrt{1+4x^2}|\Big|_0^1$



$$=\left(\tfrac{1}{2}(1)\sqrt{5}+1/4\ln|2+\sqrt{5}|\right)-\left(1/2(0)\sqrt{1}+1/4\ln|0+1|\right)$$

$$=\tfrac{1}{2}\sqrt{5}+1/4\ln|2+\sqrt{5}|\approx 1.47$$

For problems 8-10, compute the surface area for a revolution about the x-axis over the given interval.

8. $y=x$, $[0,4]$

$$S=2\pi\int_0^4 x\sqrt{1+1}\,dx$$

$$=2\pi\int_0^4 \sqrt{2}\,x\,dx$$

$$=2\pi\,\frac{\sqrt{2}}{2}x^2\Big|_0^4$$

$$=\sqrt{2}\pi x^2\Big|_0^4$$

$$\sqrt{2}\pi(16)$$

$$=16\sqrt{2}\pi$$

10. $y=(4-x^{2/3})^{3/2}$, $[0,8]$

$y' = 3/2(4-x^{2/3})^{1/2}(-2/3x^{-1/3})$

$= -x^{-1/3}(4-x^{2/3})^{1/2}$

$y'^2 = x^{-2/3}(4-x^{2/3})$

$= 4x^{-2/3}-1$

$S = 2\pi\int_0^8 (4-x^{2/3})^{3/2}\sqrt{4x^{-2/3}}\ dx$

$= 2\pi\int_0^8 (4-x^{2/3})^{3/2}\,2x^{-1/3}$ let $u=4-x^{2/3}$ $u(8)=0$

$du = -2/3\,x^{-1/3}dx$ $u(0)=4$

$= 4\pi\int_0^8 x^{-1/3}(4-x^{2/3})^{3/2}dx$ $-3/2\,du = x^{-1/3}dx$

$= -6\pi\int_4^0 u^{3/2}du$

$= -6\pi\left(\frac{2}{5}u^{5/2}\right)\Big|_4^0$

$= -6\pi(0) - (-6\pi)\left(\frac{2}{5}32\right)$

$= 6\pi(64/5) \approx 241.27$