

Math 114: Worksheet 21

Center of Mass

1. Conceptual Understanding:

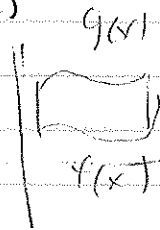
(a) Write down the formula for the coordinates of the centroid of a plate with constant density bounded between $x=a$ and $x=b$, $f(x)$, and $g(x)$ as in the figure to the right.

$$M_y = \rho \int_a^b x(g(x) - f(x)) dx$$

$$M_x = \frac{1}{2} \rho \int_a^b (g(x)^2 - f(x)^2) dx$$

$$x_{CM} = M_y / M \quad y_{CM} = M_x / M$$

$$M = \rho A = \rho \int_a^b (g(x) - f(x)) dx$$



$$y_{CM} = \frac{M_x}{M}$$

(b) Write down the formulas for the coordinates of the centroid of a plate with constant density bounded between $y=c$, $y=d$, $f(y)$, and $g(y)$ as in the figure to the right.

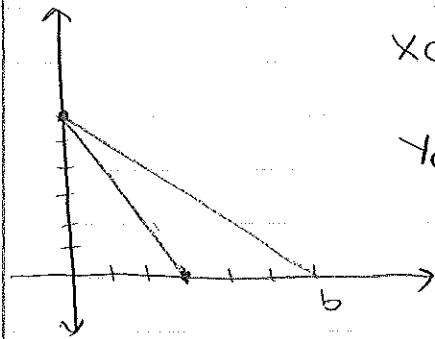
$M_x = \rho \int_c^d y(f(y) - g(y)) dy$

$$M_y = \frac{1}{2} \rho \int_c^d (f(y)^2 - g(y)^2) dy$$

$$x_{CM} = M_y / M \quad y_{CM} = M_x / M$$

$$M = \rho A = \rho \int_c^d (f(y) - g(y)) dy$$

3. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3,0)$, $(b,0)$, and $(0,b)$, where $b > 3$. Find the center of mass.

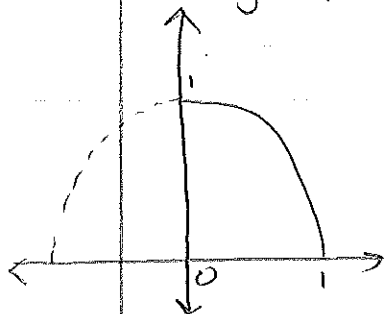


$$x_{CM} = \frac{3+b+0}{3} = \frac{3+b}{3}$$

$$y_{CM} = \frac{0+0+b}{3} = 2$$

$$COM = \left(\frac{3+b}{3}, 2 \right)$$

4. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \leq x \leq 1$.



$$\begin{aligned} M_y &= p \int_0^1 x(1-x^2) dx \\ &= p \int_0^1 x - x^3 dx \\ &= p \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 \right) \Big|_0^1 \\ &= p \left[\frac{1}{2} - \frac{1}{4} \right] \\ &= \frac{1}{4}p \end{aligned}$$

$y = 1 - u$
let $u = 1 - y$
 $du = -dy$

$$\begin{aligned} M_y &= p \int_0^1 y \sqrt{1-y} dy \\ &= -p \int_0^1 (1-u) u^{1/2} du \\ &= -p \int_0^1 u^{1/2} - u^{3/2} du \\ &= -p \left(\frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right) \Big|_0^1 \\ &= -p \left(\frac{2}{3} - \frac{2}{5} \right) \end{aligned}$$

$$\begin{aligned} M_x &= \frac{p}{2} \int_0^1 1 - 2x^2 + x^4 dx \\ &= \frac{p}{2} \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^1 \\ &= \frac{p}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) \\ &= \frac{p}{2} \left(\frac{1}{3} + \frac{1}{5} \right) \\ &= \frac{4p}{15} \end{aligned}$$

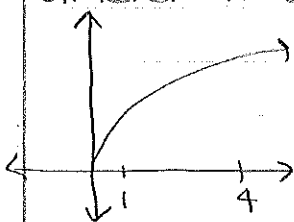
$$\begin{aligned} M &= p \int_0^1 1 - x^2 dx \\ &= p \left(x - \frac{1}{3}x^3 \right) \Big|_0^1 \\ &= p \left(1 - \frac{1}{3} \right) \\ &= \frac{2}{3}p \end{aligned}$$

$$\begin{aligned} x_{CM} &= \frac{p/4}{2p/3} \\ &= \frac{p}{4} \cdot \frac{3}{2p} \\ &= \frac{3}{8} \end{aligned}$$

$$y_{CM} = \frac{2/15 \cdot \frac{3}{8}}{2/3} = \frac{2}{5}$$

$$\boxed{\left(\frac{3}{8}, \frac{2}{5} \right)}$$

5. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.



$$\begin{aligned} M_y &= p \int_1^4 x(x^{1/2}) dx \\ &= p \int_1^4 x^{3/2} dx \\ &= p \left(\frac{2}{5} x^{5/2} \right) \Big|_1^4 \\ &= \frac{2}{5} p (32 - 1) = \frac{62}{5} p \end{aligned}$$

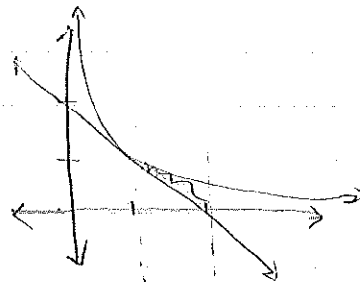
$$\begin{aligned}
 M_x &= \frac{p}{2} \int_1^4 x \, dx \\
 &= \frac{p}{2} \left(\frac{1}{2} x^2 \right) \Big|_1^4 \\
 &= \frac{p}{2} \left(8 - \frac{1}{2} \right) \\
 &= \frac{p}{2} \left(\frac{15}{2} \right) \\
 &= \frac{15p}{4}
 \end{aligned}$$

$$\begin{aligned}
 M &= p \int_1^4 \sqrt{x} \, dx \\
 &= p \left(\frac{2}{3} x^{3/2} \right) \Big|_1^4 \\
 &= p \left(\frac{16}{3} - \frac{2}{3} \right) \\
 &= \frac{14p}{3}
 \end{aligned}$$

$$\begin{aligned}
 x_{CM} &= \frac{\frac{15p}{4}}{\frac{14p}{3}} = \frac{93}{35} & y_{CM} &= \frac{\frac{15p}{4} \cdot \frac{3}{4}}{\frac{14p}{3}} = \frac{45}{56} \\
 & & & \boxed{\left(\frac{93}{35}, \frac{45}{56} \right)}
 \end{aligned}$$

* 6a. Find the centroid of the region between $f(x) = x^{-1}$ and $g(x) = 2 - x$ for $1 \leq x \leq 2$.

$$\begin{aligned}
 M_y &= p \int_1^2 x(x^{-1} - 2 + x) \, dx \\
 &= p \int_1^2 (1 - 2x + x^2) \, dx \\
 &= p \left(x - x^2 + \frac{1}{3} x^3 \right) \Big|_1^2 \\
 &= p \left((2 - 4 + \frac{8}{3}) - (1 - 1 + \frac{1}{3}) \right) \\
 &= p \left(-\frac{2}{3} + \frac{8}{3} - \frac{1}{3} \right) \\
 &= \frac{5}{3} p
 \end{aligned}$$



$$\begin{aligned}
 M_x &= \frac{p}{2} \int_1^2 (x^2 - 4 + 2x - x^2) \, dx \\
 &= \frac{p}{2} \left(-x^{-1} - 4x + x^2 - \frac{1}{3} x^3 \right) \Big|_1^2 \\
 &= \frac{p}{2} \left(\left(-\frac{1}{2} - 8 + 4 - \frac{8}{3} \right) - \left(-1 - 4 + 1 - \frac{1}{3} \right) \right) \\
 &= \frac{p}{2} \left(-\frac{3}{6} - \frac{14}{6} + \frac{2}{6} \right) \\
 &= -\frac{11}{12} p
 \end{aligned}$$

$$\begin{aligned}
 x_{CM} &= \frac{\frac{5}{3} p}{\frac{11}{12} p} = \frac{20}{11} \\
 &= \frac{1}{11} (3 \ln 2 - \frac{3}{2})
 \end{aligned}$$

$$\begin{aligned}
 M &= p \int_1^2 (x^{-1} - 2 + x) \, dx \\
 &= p \left(\ln|x| - 2x + \frac{1}{2} x^2 \right) \Big|_1^2 \\
 &= p \left(\ln 2 - 4 + 2 - 0 + 2 - \frac{1}{2} \right) \\
 &= p \left(\ln 2 - \frac{1}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 y_{CM} &= \frac{-\frac{11}{12} p}{p \left(\ln 2 - \frac{1}{2} \right)} \\
 &= \frac{1}{12 \left(\ln 2 - \frac{1}{2} \right)}
 \end{aligned}$$

7. Let $m > n > 0$. Find the centroid of the region between x^m and x^n for $0 \leq x \leq 1$. Find values for m, n that force the centroid to lie outside of the region.

$$M_y = \rho \int_0^1 x(x^n - x^m) dx$$

$$= \rho \int_0^1 x^{n+1} - x^{m+1} dx$$

$$= \rho \left(\frac{1}{n+2} x^{n+2} - \frac{1}{m+2} x^{m+2} \right) \Big|_0^1$$

$$= \rho \left(\frac{1}{n+2} - \frac{1}{m+2} \right)$$

$$\frac{m-2 - n+2}{(n+2)(m+2)}$$

$$x_{CM} = \frac{\rho \left(\frac{1}{n+2} - \frac{1}{m+2} \right)}{\rho \left(\frac{1}{n+1} - \frac{1}{m+1} \right)}$$

$$= \frac{n-m}{(m+1)(n+1)} \cdot \frac{(m+1)(n+1)}{n-m}$$

$$= \frac{(m+1)(n+1)}{(m+2)(n+2)}$$

$$M_x = \frac{\rho}{2} \int_0^1 x^{2n} - x^{2m} dx$$

$$= \frac{\rho}{2} \left(\frac{1}{2n+1} x^{2n+1} - \frac{1}{2m+1} x^{2m+1} \right) \Big|_0^1$$

$$= \frac{\rho}{2} \left(\frac{1}{2n+1} - \frac{1}{2m+1} \right)$$

$$M = \rho \int_0^1 x^n - x^m dx$$

$$= \rho \left(\frac{1}{n+1} x^{n+1} - \frac{1}{m+1} x^{m+1} \right) \Big|_0^1$$

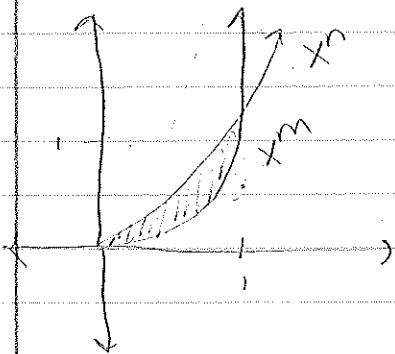
$$= \rho \left(\frac{1}{n+1} - \frac{1}{m+1} \right)$$

$$y_{CM} = \frac{\frac{\rho}{2} \left(\frac{1}{2n+1} - \frac{1}{2m+1} \right)}{\rho \left(\frac{1}{n+1} - \frac{1}{m+1} \right)}$$

$$= \frac{2n-2m}{2(2n+1)(2m+1)} \cdot \frac{(m+1)(n+1)}{n-m}$$

$$= \frac{(m+1)(n+1)}{(2m+1)(2n+1)}$$

$$= \frac{(m+1)(n+1)}{(2m+1)(2n+1)}$$



for large m, n .