Math 114: Worksheet 22

Differential Equations and $y' = k(y-b)$

1. Conceptual Understanding

(a) What does it mean to say that a differential equation is first-order?

The order of a differential equation is the order of the highest derivative appearing in the equation.

(b) What does it mean to say that a differential equation is linear or nonlinear?

A differential equation is called linear if it can be written in the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y' = b(x)$. A linear equation cannot have terms such as y^3, yy', or $\sin(y)$.

2. Use separation of variables to find the general solutions to the following differential equations:

(a) $y' + 4xy^2 = 0$

$$\frac{dy}{dx} = -4xy^2$$

$$\int \frac{1}{y^2}\,dy = \int -4x\,dx$$

$$-y^{-1} = -2x^2 + C$$

$$\frac{1}{y} = 2x^2 + C$$

$$\frac{1}{2x^2+C} = y$$

(b) $\sqrt{1-x^2}\;y' = xy$ let $u = 1-x^2$

$$\int \frac{1}{y}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx \qquad du = -2x\,dx$$

$$\ln|y| = -\tfrac{1}{2}\int u^{-1/2}\,du$$

$$\ln|y| = -(1-x^2)^{1/2} + C$$

$$y = e^{-(1-x^2)^{1/2}+C}$$

$$y = Ce^{-(1-x^2)^{1/2}}$$

(c) $(1+x^2)y' = x^3 y$

$\int \frac{1}{y}\,dy = \int x^3/(1+x^2)\,dx$ $\qquad$ $x^2+1\overset{x}{\overline{)x^3}}$ $\qquad$ let $u = 1+x^2$

$\ln|y| = \int (x^2+1)x - x/(1+x^2)\,dx$ $\qquad$ $\underline{x^3+x}$ $\qquad$ $du = 2x\,dx$

$\ln|y| = \int x\,dx - \int \frac{x}{1+x^2}\,dx$ $\qquad$ $-x$

$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\int \frac{1}{u}\,du$

$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\ln|1+x^2| + C$

$y = e^{1/2x^2 + \ln|1+x^2|^{-1/2} + c}$

$= e^{1/2x^2}(1+x^2)^{-1/2}C$

$= Ce^{1/2x^2}/\sqrt{1+x^2}$

(d) $\sqrt{1+y^2}\,y' + \sec x = 0$ $\qquad$ let $y = \tan\theta$

$\int \sqrt{1+y^2}\,dy = \int -\sec x\,dx$ $\qquad$ $dy = \sec^2\theta$

$\int \sec^3\theta \cdot d\theta = -\ln|\sec x + \tan x|$

$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| = -\ln|\sec x + \tan x| + C$

$1/2\,y\sqrt{1+y^2} + 1/2\ln|\sqrt{1+y^2} + y| = -\ln|\sec x + \tan x| + C$

3. Solve $y' = 4y + 24$ subject to the condition to that $y(0) = 5$.

$\int \frac{1}{y+6}\,dy = \int 4\,dx$

$\ln|y+6| = 4x + C$

$y + 6 = e^{4x+c}$

$y = Ce^{4x} - 6$, use initial conditions

$5 = Ce^{0} - 6$

$11 = C$

$y = 11e^{4x} - 6$

4. Solve $y' + 6y = 12$ subject to the condition that $y(2) = 10$.

$y' = 6(2-y)$

$\int \frac{1}{2-y}\,dy = \int 6\,dx$

$\ln|2-y| = -6x + C$

$2 - y = Ce^{-6x}$

$2 - Ce^{-6x} = y$

$2 - Ce^{-12} = 10$

$-Ce^{12}=8$

$C=-8e^{12}$

$y=2+8e^{12-6k}$

5. $y'=-k(y-T_0)$, k = constant depending on the object, T_0 = temp. of the ambient environment Frank's car engine runs at 210° F. On a 70° F day, he turns off the ignition and notes that five minutes later, the engine has cooled to 160°F.

(a) Find the cooling constant k.

$\int \frac{1}{y-70°}\,dy=\int -k\,dt$

$\ln|y-70°|=-kt+C$

$y-70°=Ce^{-kt}$

$y=Ce^{-kt}+70$

$210°=Ce^{0}+70°$

$140=C$

$160=140e^{-5k}+70$

$90=140e^{-5k}$

$\frac{9}{14}=e^{-5k}$

$\ln|\frac{9}{14}|=-5k$

$\boxed{k=-1/5\ln(9/4)}$

≈ 08836

(b) when will the engine cool to 100° F?

$100=140e^{\ln(9/4)^{-1/5t}}+70$

$30=140(9/4)^{-t/5}$

$\ln(3/4)=-t/5(\ln(9/4))$

$\frac{\ln(3/4)}{\ln(9/4)}=-t/5$

$\frac{5\ln(3/4)}{\ln(9/4)}=t\approx 17:432$ should be 17.432

6. A cup of coffee with cooling constant $k=0.09\,\text{min}^{-1}$ is placed in a room of temperature 20°C.

(a) How quickly is the coffee cooling when the temperature is 80°C?

$\frac{dy}{dt}=-k(y-20)$

$\frac{dy}{dt}=-0.09(80-20)=-0.09(60)=-5.4$

(b) Use the linear approximation to estimate the change in temperature over the next 6s when the temperature is 80°C.

$(\frac{1}{10})(-5.4) = -.54$

(c) If the coffee is initially served at 90°c, how long will it take to reach an optimal drinking temperature of 65°c.

$y = 20 + Ce^{-kt}$

$90 = 20 + Ce^{0}$

$70 = C$

$65 = 20 + 70e^{-.09t}$

$45 = 70e^{-.09t}$

$45/70 = e^{-.09t}$

$\ln|45/70| = -.09t$

$t = \frac{1}{-.09}\ln|45/70| \approx 4.9$ min.

7. $\frac{dy}{dt} = \frac{Bv(y)}{A(y)}$ $\quad v(y) = -\sqrt{19.6y}$ $\quad A(y) = \pi(\sqrt{y})^2 = \pi y$

$\frac{dy}{dt} = \frac{-0.0005\sqrt{19.6y}}{\pi y} \rightarrow \sqrt{y}\,dy = \frac{-0.0005\sqrt{19.6}}{\pi}dt$

$2/3\, y^{3/2} = \frac{-0.0005\sqrt{19.6}}{\pi}t + C$

At $y(0), 1$

$2/3 = C$

$2/3\, y^{3/2} = \frac{-0.0005\sqrt{19.6}}{\pi}t + 2/3$

$0 = \frac{-0.0005\sqrt{19.6}}{\pi}t + 2/3$

$-2/3(\pi) = -0.0005\sqrt{19.6}\,t$

$t \approx 946.15$