Math 114: Worksheet #24

Review for Exam 3

1. Compute

(a) $\int \frac{dx}{x^2-6x+8}$ $\qquad \frac{1}{(x-4)(x-2)} = \frac{A}{(x-4)} + \frac{B}{(x-2)}$

$= \int \frac{dx}{(x-4)(x-2)}$ $\qquad 1 = (x-2)A + (x-4)B$

$x=2, \Rightarrow B = -\frac{1}{2}, \quad x=4 \Rightarrow A = 1/2$

$= \frac{1}{2}\int \frac{1}{x-4}\,dx - \frac{1}{2}\int \frac{1}{x-2}\,dx$

$= \frac{1}{2}\ln|x-4| - \frac{1}{2}\ln|x-2| + C$

(b) $\int \frac{3}{(x+1)x(x+1)}\,dx$ $\qquad \frac{3}{(x+1)^2 x} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x}$

$= \int \frac{3}{x(x+1)^2}\,dx$ $\qquad 3 = x(x+1)A + xB + (x+1)^2 C$

$x=0, \Rightarrow C = 3$

$= -3\int \frac{dx}{(x+1)}$ $\qquad x=-1, \Rightarrow B = -3$

$x = 1, \Rightarrow 3 = 2A - 3 + 12$

$-3\int \frac{dx}{(x+1)^2}$ $\qquad \Rightarrow -6 = 2A, \ -3 = A$

$+3\int \frac{dx}{x} \quad = -3\ln|x+1| + 3(x+1)^{-1} + 3\ln|x| + C$

(c) $\int \frac{x^2}{x^2+9}\,dx$ $\qquad$ let $x = 3\tan\theta \quad x^2+9 = 9\sec^2\theta$

$dx = 3\sec^2\theta$

$= \int \frac{9\tan^2\theta\, 3\cancel{\sec^2}\theta\, d\theta}{9\cancel{\sec^2}\theta}$

$= \int 3\tan^2\theta\, d\theta$

$= \int 3\sec^2\theta - 3\, d\theta$

$= 3\tan\theta - 3\theta + C$

$= 3\tan\theta \ldots - \tan^{-1}(\frac{x}{3}) + C$

$= x^2/3 - \tan^{-1}(\frac{x}{3}) + C$

(d) $\int \frac{x^2+2}{x+3}\,dx$

$$\begin{array}{r} x-3 \\ x+3\,\overline{)\,x^2+0x+2} \\ -(x^2+3x) \\ \hline -3x+2 \\ -(-3x-9) \\ \hline 11 \end{array}$$

$= \int \frac{(x+3)(x-3)+11}{x+3}\,dx$

$= \int x-3\,dx + 11\int \frac{dx}{x+3}$

$= \frac{1}{2}x^2 - 3x + 11\ln|x+3| + C$

2. Compute $\int \frac{e^x}{e^{2x}-e^x}\,dx$.

Let $u = e^x$, $du = e^x dx$

$\int \frac{du}{u^2-u} = \int \frac{du}{u(u-1)}$ $\qquad \frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$= \int \frac{-du}{u} + \int \frac{du}{u-1}$ $\qquad 1 = (u-1)A + uB$

$u=0, \Rightarrow A=-1$

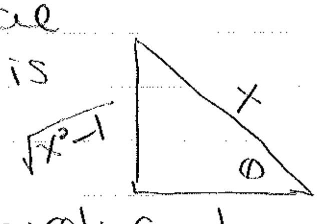
$u=1, \Rightarrow B=1$

$= -\ln|u| + \ln|u-1| + C$

$= \ln|e^x-1| - \ln|e^x| + C$

$= \ln|e^x-1| - x + C$

3. Evaluate $\int \frac{dx}{x^2-1}$ first with a trig substitution and then with partial fractions. Verify that the answer is the same in both cases.



Let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\int \frac{\sec\theta\,\cancel{\tan}\theta}{\tan^{\cancel{2}}\theta}\,d\theta$

$= \int \frac{\sec\theta}{\tan\theta}\,d\theta$

$= \int \frac{1}{\sin\theta}\,d\theta$

$= \int \csc\theta\,d\theta$

$= -\ln|\csc\theta + \cot\theta| + C$

$= -\ln\left|\frac{x}{\sqrt{x^2-1}} + \frac{1}{\sqrt{x^2-1}}\right| + C$

$= -\ln\left|\frac{x+1}{\sqrt{x^2-1}}\right| + C$

$= -\ln|x+1| + \ln|\sqrt{x^2-1}| + C$

$= -\ln|x+1| + \frac{1}{2}\ln|(x-1)(x+1)| + C$

$= -\ln|x+1| + \frac{1}{2}\ln|(x-1)|$
$+ \frac{1}{2}\ln|x+1| + C$

$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C_1$

$$\int \frac{1}{(x+1)(x-1)}\,dx \qquad \frac{1}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$$

$$= -\frac{1}{2}\int \frac{dx}{(x+1)} \qquad 1 = (x-1)A + (x+1)B$$

$$x = 1,\ B = \frac{1}{2}$$

$$+\frac{1}{2}\int \frac{dx}{(x-1)} \qquad x = -1,\ A = -\frac{1}{2}$$

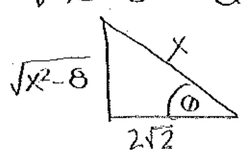
$$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

4. Use trigonometric substitution to evaluate the integral $\int \frac{dx}{x^2\sqrt{x^2-8}}$

Let $x = 2\sqrt{2}\sec\theta$

$dx = 2\sqrt{2}\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-8} = 2\sqrt{2}\tan\theta$



$$= \int \frac{\cancel{2\sqrt{2}}\sec\theta\cancel{\tan\theta}\,d\theta}{8\sec^2\theta\,\cancel{2\sqrt{2}\tan\theta}}$$

$$= \frac{1}{8}\int \cos\theta\,d\theta$$

$$= \frac{1}{8}\sin\theta + C$$

$$= \frac{1}{8}\left(\sqrt{x^2-8}/x\right) + C$$

5. Find the arc length of $f(x) = \ln(\sec(x))$ from $x = 0$ to $x = \pi/4$

$f'(x) = \frac{1}{\sec(x)}(\sec(x)\tan(x))$

$f'(x) = \tan x$

$[f'(x)]^2 = \tan^2 x$

$$S = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx$$

$$= \int_0^{\pi/4} \sec x\,dx$$

$$= \ln|\sec x + \tan x|\Big|_0^{\pi/4}$$

$$= \ln|\sec(\pi/4) + 1| - \ln|\cancelto{0}{1+0}|$$

$$= \ln|\sqrt{2} + 1|.$$

6. Find the surface area of the solid of revolution obtained by revolving $\sqrt{9-x^2}$ about the x-axis for $-2 \le x \le 2$.

$$f'(x) = \frac{1}{2}(9-x^2)^{-1/2}(-2x)$$
$$= -x/\sqrt{9-x^2}$$
$$[f'(x)]^2 = x^2/9-x^2$$

$$S = 2\pi \int_{-2}^{2} \sqrt{9-x^2}\sqrt{1+\frac{x^2}{9-x^2}}\, dx$$

$$= 4\pi \int_0^2 \sqrt{9-x^2}\sqrt{9/9-x^2}\, dx$$

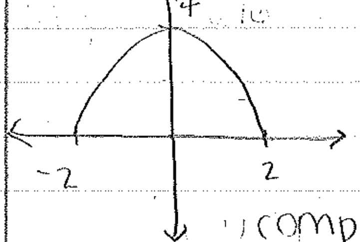
$$= 4\pi \int_0^2 3\, dx$$

$$= 12\pi (x)\Big|_0^2$$
$$= 24\pi$$

7. Consider point masses m_1, m_2, and m_3 centered at $(-1,0)$, $(3,0)$, and $(0,4)$ respectively. If $m_1 = 6$, find m_2 so that the center of mass lies on the y-axis.

$$0 = 6(-1) + m_2(3) + m_3(0) = M_y$$
$$0 = -6 + 3m_2$$
$$6 = 3m_2$$
$$2 = m_2$$

8. Find the centroid of the top half of the ellipse $(\frac{x}{2})^2 + (\frac{y}{4})^2 = 1$.



the region is symmetric with respect to the y-axis, so we know in advance that $x_{CM} = 0$. To find y_{CM}, we compute the moment M_x.

$$\frac{y^2}{16} = \frac{4-x^2}{4}$$

$$y^2 = \frac{64 - 16x^2}{4}$$

$$y = \sqrt{16 - 4x^2}$$

$$M_x = \frac{1}{2}\rho\int_{-2}^{2} f(x)^2 dx$$

$$= \rho\int_0^2 16 - 4x^2 dx$$

$$= \rho(16x - \frac{4}{3}x^3)\Big|_0^2$$

$$= \rho(64/3)$$

$$M = 2\rho\int_0^2 \sqrt{16-4x^2}\, dx \qquad \text{let } x = 2\sin\phi$$

$$= 2\rho\int_0^2 8\cos^2\phi\, d\phi \qquad dx = 2\cos\phi\, d\phi$$

$$= 16\rho\int_0^2 \cos^2\phi\, d\phi \qquad \sqrt{16-4x^2} = 4\cos\phi$$

$$= 8\rho\int_0^2 1 + \cos 2\phi\, d\phi$$

$$= 8\rho(\phi + \sin\phi\cos\phi)\Big|_0^2$$

$$= 8\rho(\sin^{-1}(\frac{x}{2}) + \frac{x}{8}\sqrt{16-4x^2})\Big|_0^2$$

$$= 8\rho(\frac{\pi}{2})$$

$$= 4\rho\pi$$

$$\frac{M_x}{M} = \frac{\rho(\frac{64}{3})}{\rho\, 4\pi/1}$$

$$= 64/3 \cdot 1/4\pi$$

$$\boxed{= 16/3\pi}$$

centroid: $(0, 16/3\pi)$.

9. Use separation of variables to solve $y' + 4xy^2 = 0$.

$$y' = -4xy^2$$

$$\int \frac{y'}{y^2} dx = \int -4x\, dx$$

$$-y^{-1} = -2x^2 + C$$

$$y^{-1} = 2x^2 + C$$

$$y = \frac{1}{2x^2 + C}$$

10. Use separation of variables to solve

$y' = (x+1)(y^2+1)$

$\int \frac{1}{y^2+1}dy = \int (x+1)dx$

$\arctan(y) = \frac{1}{2}x^2 + x + C$

$y = \tan(\frac{1}{2}x^2 + x + C)$

11. Solve the initial value problem

$y' + y^2 \sin x = 0, \; y(\pi) = 2$

$y' = -y^2 \sin x$

$\int \frac{1}{y^2}dy = \int -\sin x \, dx$

$-y^{-1} = \cos x + C$

$y^{-1} = -\cos x + C$

$y = \frac{1}{-\cos x + C}$

$2 = \frac{1}{1+C}$

$2 + 2C = 1$

$2C = -1$

$C = -1/2$

12. Find the solutions to $y' = -2y + 8$ subject to $y(0) = 3$ and $y(0) = 4$, respectively, and sketch their graphs.

$y' = -2(y-4)$

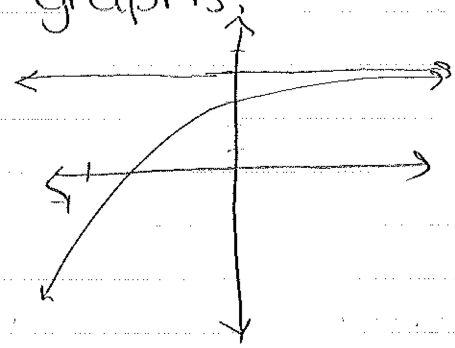
$\int \frac{1}{y-4}dy = \int -2dx$

$\ln|y-4| = -2x + C$

$y - 4 = e^{-2x+C}$

$y - 4 = Ce^{-2x}$

$y = Ce^{-2x} + 4$



$y(0) = 3$

$3 = C + 4$

$-1 = C$

$y = -e^{-2x} + 4$

$y(0) = 4$

$4 = C + 4$

$0 = C$

$y = 4$