$$y = \frac{A}{1-e^{-kt}/C}$$

Math 114: Worksheet #25: The Logistic Equation and First-order Linear Equations.

1. The population of the world in 1990 was around 5.3 billion. Assume the growth constant is 1/265 and the carrying capacity is 100 billion.

(a) Write out the logistic model and solve it.

$$dy/dt = \frac{1}{265} y \left(1 - y/100\right), \quad y(0) = 5.3$$

$$y = \frac{100}{1-e^{-\frac{1}{265}t}/C} \qquad \boxed{y = \frac{100}{1-e^{-\frac{1}{265}t}/-0.0559}}$$

$$5.3 = \frac{100}{1-\frac{1}{C}}$$

$$5.3 - 5.3/C = 100$$

$$-5.3/C = 94.7$$

$$-5.3 = 94.7C$$

$$-0.0559 = C$$

(b) Use this to estimate the population in 2014 and compare it with the actual population of 7.2 billion.

$t = 24$.

$$y = \frac{100}{1-e^{-24/265}/-0.0559}$$

$\boxed{= 5.76669}$ billion, smaller than the actual population.

(c) Use the logistic model to predict the population in 2100 and 2500

$t = 110$

$$y = \frac{100}{1-e^{-110/265}/-0.0559}$$

$\boxed{= 7.91 \text{ billion}}$

$t = 510$

$$y = \frac{100}{1-e^{-510/265}/-0.0559}$$

$\boxed{= 27.69 \text{ billion}}$

2. Assume the carrying capacity of the U.S. population is 5 billion.

(a) Use this and the fact that the population in 1990 was 250 million to find the logistic model for the U.S. population.

$\frac{dy}{dt} = ky\left(1-\frac{y}{5}\right)$ $\quad y(0)=250$

turn billion to million, or visa versa.

$y = \frac{500}{1-e^{-kt}/C} \rightarrow 250 = \frac{5000}{1-1/C}$

$250 - 250/C = 5000$

$-250 = 250/C$

$C = -1 \approx -.05263$

(b) Use the fact that the population in 2000 was 275 million to find k and P(t).

$275 = \frac{5000}{1+e^{-10k}/-.05263}$

$275 + 5225e^{-10k} = 500$

$275e^{-10k} = 225$

$e^{-10k} = .81818$

$-10k = -.200667$

$\boxed{k = .0200667} \longrightarrow .0100587$

$\boxed{P(t) = \frac{500}{1+e^{.0200667t}}}$ $\quad P(t) = \frac{5000}{1+\frac{e^{-.0100587t}}{.05263}}$

(c) Predict the U.S. population in 2100 and 2500.

$t = 110$ $\quad t = 510$

P(t) = ~~49.5~~ million. $\quad$ P(t) = ~~.01796~~ million.

~~85~~.5 million $\quad$ ~~50.775~~ million.

683 mill $\quad$ 4.503 billion

* 3. A lake with a carrying capacity of 10,000 fish is stocked with 400 fish. The number of fish triples in the first year.

(a) Find the logistic model and solve it. (Also find k).

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{10{,}000}\right)$$

$$y = \frac{10{,}000}{1 - e^{-kt}/c} \qquad y(0) = 400$$

$$400 = \frac{10{,}000}{1 - 1/c}$$

$$400 - 400/c = 10{,}000$$

$$-400/c = 9{,}600$$

$$c = -.041666$$

$$1/c = -24$$

$$y = \frac{10{,}000}{1 + 24e^{-kt}}$$

$$y(1) = 1200$$

$$1200 = \frac{10{,}000}{1 + 24e^{-k}}$$

$$1200 + 28800e^{-k} = 10{,}000$$

$$28800e^{-k} = 8800$$

$$e^{-k} = .30555$$

$$-k = -1.1856$$

$$\boxed{k = 1.1856}$$

(b) How long will it take for the population to reach 5000 fish?

$$5000 = \frac{10{,}000}{1 - e^{-1.1856t} \cdot 24}$$

$$5000 + 208.33e^{-1.1856t} = 10{,}000 \qquad -1 = -.041666\, e$$

$$208.33e^{-1.1856t} = 5000$$

$$e^{-1.1856t} = 24$$

$$-1.1856t = 3.178$$

$$t = 2.6\text{ yrs}$$

4. Let $c>0$. A differential equation of the form $dy/dt = ky^{1+c}$, $k>0$ is called a doomsday equation because $1+c>1$.

(a) Use separation of variables to find the solution of this model with $y(0)=y_0$.

$\int y^{\frac{1}{1+c}} dy = \int k\,dt$

$-\frac{1}{c}y^{-c} = kt + C$

$y(0)=y_0$

$-\frac{1}{c}y_0^{-c} = C$

$\boxed{-\frac{1}{c}y^{-c} = kt - \frac{1}{c}y_0^{-c}}$ $\rightarrow$ $y^{-c} = -ckt + y_0^{-c}$

$y = (-ckt + y_0^{-c})^{-1/c}$

(b) Show that there is a finite time $t=T$ (doomsday) such that $\lim_{t\to T} y(t) = \infty$

$\lim_{t\to T}(-ckt + y_0^{-c})^{-1/c}$ Note that c is positive, which means that $-1/c$ is negative. Taking zero to an infinite power results in an infinite discontinuity (i.e. doomsday).

$-ckt + y_0^{-c} = 0$

$y_0^{-c} = ckt$

$t = y_0^{-c}/ck$, this t is the doomsday.

(c) A certain breed of rabbits has the growth rate term $ky^{1.01}$. Suppose the initial population is 2 and there are 16 rabbits after 3 months. When is doomsday?

In this case, $c = 0.01$, $y_0 = 2$. $y(3)=16$.

$16 = (-0.01k(3) + 2^{-0.01})^{-1/0.01}$

$k = .681$

$t = 2^{-.01}/.01(.681) \approx \boxed{145.8285}$

5. Consider $y' + x^{-1}y = x^3$.

(a) Verify that $\alpha(x) = x$ is an integrating factor,

$$\alpha(x) = e^{\int x^{-1}dx}$$
$$= e^{\ln|x|}$$
$$= x$$

(b) Show that when multiplied by $\alpha(x)$, the differential equation can be written as $(xy')=x^4$

$$xy' + y = x^4$$
$$(xy)' = x^4$$

* product rule*

(c) Conclude that xy is an antiderivative of x^4 and use this information to find the general solution.

$$\int (xy)' = \int x^4\,dx$$
$$xy = \tfrac{1}{5}x^5 + C$$
$$y = \tfrac{1}{5}x^4 + x^{-1}C$$

(d) Find the particular solution satisfying $y(1)=0$.

$$0 = \tfrac{1}{5}(1)^4 + (1)^{-1}C$$
$$0 = \tfrac{1}{5} + C$$
$$-\tfrac{1}{5} = C$$
$$\boxed{y = \tfrac{1}{5}x^4 - \tfrac{1}{5}x^{-1}}$$

6. Solve the following differential equations.

(a) $xy' = y - x$, $y' = \frac{y}{x} - 1$, $y' + A(x)y = B(x)$

$\alpha(x) = e^{\int -x^{-1}dx}$ $\quad -xy' - y = \frac{1}{x}$ $\quad y = -x\ln|x| - Cx$

$= e^{-\ln|x|}$ $\quad -\int(\frac{1}{x}y)' = \int \frac{1}{x}$

$= \frac{1}{x}$ $\quad -\frac{1}{x}y = \ln|x| + C$

(b) $y' + 3x^{-1}y = x + x^{-1}$

$\alpha(x) = e^{\int 3/x\,dx}$

$= e^{3\ln x}$

$= e^{\ln x^3}$

$= x^3$

$y = x^{-3}\int x^3(x + x^{-1})\,dx$

$= x^{-3}\int x^4 + x^2\,dx$

$= x^{-3}\left(\frac{1}{5}x^5 + \frac{1}{3}x^3 + C\right)$

$= \frac{1}{5}x^2 + \frac{1}{3} + Cx^{-3}$

7. Solve the following differential equations that satisfy the initial condition:

(a) $y' + 3y = e^{2x},\ y(0) = -1$

$\alpha(x) = e^{\int 3\,dx}$

$= e^{3x}$

$y = e^{-3x}\left(\int e^{3x}e^{2x}\,dx\right)$

$= e^{-3x}\left(\frac{1}{5}e^{5x} + C\right)$

$= \frac{1}{5}e^{2x} + Ce^{-3x}$

$-1 = \frac{1}{5}(1) + C(1)$

$-\frac{6}{5} = C$

$y = \frac{1}{5}e^{2x} - \frac{6}{5}e^{-3x}$

(b) $(\sin x)y' = (\cos x)y + 1,\ y(\pi/4) = 0.$

$y' = \left(\frac{\cos x}{\sin x}\right)y + \frac{1}{\sin x}$

$\alpha(x) = e^{\int \frac{\cos x}{\sin x}dx}$

$= e^{\ln|\sin x|}$

$= \sin(x)$

$y = \frac{1}{\sin(x)}\int \sin(x)\left(\frac{1}{\sin x}\right)dx$

$= \frac{1}{\sin(x)}\int dx$

$= \frac{1}{\sin(x)}(x + C)$

$= \frac{x}{\sin(x)} + C/\sin(x)$

$0 = \frac{\pi/4}{\sqrt{2}/2} + \sqrt{2}C$

$0 = \sqrt{2}\pi/2 + \sqrt{2}C$

$-\sqrt{2}\pi/2 = \sqrt{2}C$

$-\pi/2 = C$

$y = \frac{x}{\sin(x)} - \frac{\pi}{2\sin(x)}$