Math 114: Worksheet #26: First-Order Linear Equations and Parametric Equations

1. Solve the second order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

$$xu' + 2u = 12x^2$$
$$u' + \frac{2}{x}u = 12x$$
$$\alpha(x) = e^{\int \frac{2}{x} dx}$$
$$= e^{2\ln|x|}$$
$$= x^2$$

$$u = x^{-2}\int x^2 12x \, dx$$
$$= x^{-2}\int 12x^3 \, dx$$
$$= x^{-2}\left(\frac{12}{4}x^4 + C\right)$$
$$= 3x^2 + Cx^{-2}$$
$$y = \int 3x^2 + Cx^{-2} \, dx$$
$$y = x^3 - Cx^{-1} + D.$$

2. Consider a series circuit consisting of a resistor of R ohms, an inductor of L henries and a variable voltage source of $V(t)$ volts. The current through the circuit $I(t)$ satisfies the differential equation $dI/dt + R/L\, I = \frac{1}{L}V(t)$. Assume that $R = 110\Omega$, $L = 10H$, and $V(t) = e^{-t}$ volts.

(a) Solve the equation with initial condition $I(0) = 0$.

$$dI/dt + 110/10\, I = \frac{1}{10}e^{-t}$$
$$\alpha(x) = e^{\int 11 dt}$$
$$= e^{11t}$$
$$I = e^{-11t}\left(\int e^{11t}\left(\frac{1}{10}e^{-t}\right)dt\right)$$
$$= e^{-11t}\int \frac{1}{10}e^{10t} dt$$
$$= e^{-11t}\left(\frac{1}{100}e^{10t} + C\right)$$
$$= \frac{e^{-t}}{100} + Ce^{-11t}$$

$$I(0) = 0$$
$$0 = \frac{1}{100} + Ce^0$$
$$0 = 1 + C$$
$$\frac{-1}{100} = C$$

$$I = \frac{e^{-t}}{100} - \frac{1}{100}e^{-11t}$$

$$\frac{dI}{dt} = \frac{-11e^{-t}}{100} + \frac{11}{100}e^{-11t} + \frac{10}{100}e^{-t}$$

$$0 = \frac{-e^{-t}}{100} + \frac{11}{100}e^{-11t} \qquad 0 = e^{-t}\left(-\frac{1}{100} + \frac{11}{100}e^{-10t}\right)$$

$$\frac{1}{100} = \frac{11}{100}e^{-10t}$$

(b) Calculate t_m and $I(t_m)$, where t_m is the time at which $I(t)$ has maximum value.

$$dI/dt = -e^{-t} + 11e^{-11t} = 0$$

$$11e^{-11t} = e^{-t}$$

$$\ln(11) + \ln(e^{-11t}) = \ln(e^{-t})$$

$$\ln(11) - 11t = -t$$

$$\ln(11) = 10t$$

$$\boxed{\ln(11)/10 = t}$$

$$\frac{1}{11} = e^{-10t}$$

$$\ln\frac{1}{11} = -10t$$

$$\frac{\ln(\frac{1}{11})}{-10} = t$$

$$I\left(\frac{\ln(11)}{10}\right) = e^{-\ln(11)/10} - e^{-11(\ln(11)/10)} = e^{\ln(11)^{-1/10}} - e^{\ln(11)^{-11/10}}$$

$$= 11^{-1/10} - 11^{-11/10} \approx$$

3. A tank with a capacity of 400 liters is full of a mixture of water and chlorine with a concentration of 0.05 grams of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 liters per second. The mixture is kept stirred and is pumped out at a rate of 10 liters per second. Find the amount of chlorine in the tank as a function of time.

$$dy/dt = \underbrace{\text{chlorine rate in}}_{0\,kg/L\,(4\,L/s)} - \underbrace{\text{chlorine rate out}}$$

concentration at time t: $y(t)/400-6t$ kg/L

$y(t)$ = kg of chlorine in tank

$$\text{chlorine rate out} = \left(\frac{y(t)}{400-6t}\ \frac{kg}{L}\right)\left(10\ \frac{L}{s}\right)$$

$$= \frac{10y}{400-6t}$$

$$\frac{dy}{dt} = \frac{-10y}{400-6t}$$

$$\int \frac{dy}{10y} = \int \frac{-dt}{400-6t}$$

$$\frac{\ln|10y|}{10} = \frac{\ln|400-6t|}{6} + C$$

$$\ln|10y| = \tfrac{5}{3}\ln|400-6t| + C$$

$$10y = C(400-6t)^{5/3}$$

$$y = C(400-6t)^{5/3}$$

$$y(0) = 20$$

$$20 = C(400)^{5/3}$$

$$C \approx .000921$$

$$y = .000921(400-6t)^{5/3}$$

4. Conceptual Understanding:

(a) How is a curve different from a parametrization of the curve?

A parametrization of a curve not only describes the underlying curve, but a particular way of moving along the curve.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

(b). Suppose a curve is parametrized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

there is a vertical tangent at t_0 and the curve is concave up around t_0.

5. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, (\frac{t}{\pi})^2)$, for $0 \le t \le 2\pi$.

(a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

$.9571$

$c(0) = (0, 0)$

$c(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2} + \frac{1}{4}, \frac{1}{16})$

$c(\frac{\pi}{2}) = (1 + \frac{1}{2}, \frac{1}{4})$

$c(\frac{3\pi}{4}) = (\frac{\sqrt{2}}{2} + \frac{3}{4}, \frac{9}{16})$

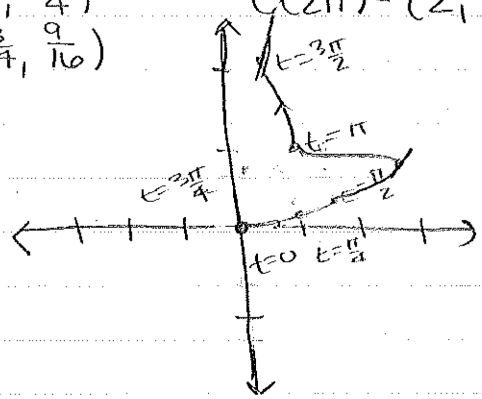
$\approx .04289$

$t = 2\pi$

$c(\pi) = (1, 1)$

$c(\frac{3\pi}{2}) = (-1 + \frac{3}{2}, \frac{9}{4})$

$c(2\pi) = (2, 4)$



(b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \pi/2$ and $t = 3\pi/2$. What does this tell you about the curve near these points?

$x'(t) = \cos(t) + \frac{1}{\pi}$

$x'(\frac{\pi}{2}) = \frac{1}{\pi}$

$x'(\frac{3\pi}{2}) = \frac{1}{\pi}$

$y'(t) = 2(\frac{t}{\pi^2})$

$y'(\frac{\pi}{2}) = \frac{1}{\pi}$

$y'(\frac{3\pi}{2}) = 3/\pi$

the slope

(c) Use the above information to plot the curve.

6. (a) $x=\sqrt{t}$, $y=1-t$

$t=1-y$

$x=\sqrt{1-y}$

$x^2=1-y$

$\boxed{y=1-x^2}$

(b) $x=3t-5$, $y=2t+1$

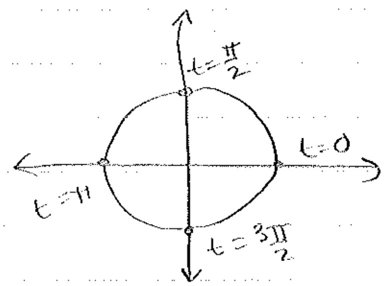
$x+5=3t$ $\qquad$ $y=\frac{2}{3}(x+5)+1$

$\frac{x+5}{3}=t$ $\qquad$ $y=\frac{2}{3}x+\frac{10}{3}+\frac{3}{3}$

$y=\frac{2}{3}x+\frac{13}{3}$

(c) $x=\cos t$, $y=\sin t$

$x^2+y^2=1$



7. $y=x^3$ from $x=0$ to $x=2$

(a) Let $y=t$ be the parameter

$y=t$ $\qquad$ $0\le t\le\sqrt[3]{2}$

$x=\sqrt[3]{t}$

(b) $\frac{x^2}{4}+\frac{y^2}{9}=1$

Equation of an ellipse

$x=2\cos t$ $\qquad$ $-\pi\le t<\pi$

$y=3\sin t$.

8. A particle travels from the point (2,3) to (-1,-1) along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time t.

$x = 2 + t \qquad 0 \le t \le 5$

$y = -1 - 2t$

$x = \frac{3}{5}t - 1$

$y = \frac{4}{5}t - 1$

$(3t-1, 4t-1$