Math 114: Worksheet #27

1. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.

(a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$

pt $(e, 1)$ $\quad x' = \frac{1}{2}t^{-1/2}e^{\sqrt{t}}$ $\quad y' = 1 - \frac{1}{t^2}(2t)$

$= 1 - \frac{2}{t}$

$$\text{slope} = \frac{1 - \frac{2}{t}}{\frac{1}{2}t^{-1/2}e^{\sqrt{t}}} = \frac{1-2}{\frac{1}{2}e} = \frac{-1}{\frac{1}{2}e} = \frac{-2}{e}$$

$1 = -\frac{2}{e}(e) + b$

$1 = -2 + b$

$3 = b$

$y = -\frac{2}{e}x + 3$

(b) $x = \cos(\theta) + \sin(2\theta)$, $y = \sin(\theta)$ at $\theta = \pi/2$

pt $(0, 1)$ $\quad x' = -\sin\theta + 2\cos(2\theta)$ $\quad y' = \cos\theta$

$$\text{slope} = \frac{\cos\theta}{-\sin\theta + 2\cos\theta} = 0$$

$y = 1$

2. For the following parametric curve, find $dy/dx$.

(a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$

$x' = \frac{1}{2}t^{-1/2}e^{\sqrt{t}}$ $\quad y' = 1 - e^{-t}$

$$\frac{dy}{dx} = \frac{1 - e^{-t}}{\frac{1}{2}t^{-1/2}e^{\sqrt{t}}}$$

(b) $x = t^3 - 12t$, $y = t^2 - 1$

$x' = 3t^2 - 12$ $\quad y' = 2t$

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 12}$$

(c) $x = 4\cos(t)$, $y = \sin(2t)$

$x' = -4\sin(t)$ $\quad y' = 2\cos(2t)$

$$\frac{dy}{dx} = \frac{2\cos(2t)}{-4\sin(t)}$$

3. Find the arc length of the following curves.

(a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \le t \le 1$.

$$S = \int_0^1 \sqrt{[6t]^2 + [6t^2]^2}\, dt$$

$$= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_0^1 \sqrt{36t^2(1+t^2)}\, dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\, dt \qquad \begin{aligned} u &= 1+t^2 \\ du &= 2t\,dt \\ 3du &= 6t\,dt \end{aligned}$$

$$= \int_1^2 3\sqrt{u}\, du$$

$$= \frac{2}{3}(3)u^{3/2}\Big|_1^2 = 2u^{3/2}\Big|_1^2 = 2(2\sqrt{2}) - 2 = 4\sqrt{2} - 2$$

(b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \le t \le 2\pi$

$$S = \int_0^{2\pi} \sqrt{(-4\sin(t))^2 + (4\cos(t))^2}\, dt$$

$$= \int_0^{2\pi} 4\,dt$$

$$= 4t\Big|_0^{2\pi} = 8\pi$$

(c) $x=3t^2,\ y=4t^3,\ 1\le t\le 3$

$$s=\int_1^3\sqrt{(6t)^2+(12t^2)^2}\,dt$$

$$=\int_1^3\sqrt{36t^2+144t^4}\,dt$$

$$=\int_1^3\sqrt{36t^2(1+4t^2)}\,dt$$

$$=\int_1^3 6t\sqrt{1+4t^2}\,dt \qquad \begin{aligned} u&=1+4t^2\\ du&=8t\,dt\cdot\tfrac{3}{4}\\ \tfrac{3}{4}du&=6t\,dt\end{aligned}$$

$$=\int_5^{37}\tfrac{3}{4}\sqrt{u}\,du$$

$$=\tfrac{2}{3}\left(\tfrac{3}{4}\right)u^{3/2}\Big|_5^{37}=\tfrac{1}{2}u^{3/2}\Big|_5^{37}=\tfrac{1}{2}(37)^{3/2}-\tfrac{1}{2}(5)^{3/2}$$

$$\approx 106.94$$

6. Consider the line through $P=(1,0)$ and $Q=(7,8)$. Find the parametrizations of this line with the following speeds.

$(6at+1,\ 8at)$

$x'=6a$

$y'=8a$

(a) $s'(t)=1=\sqrt{36a^2+64a^2}$

$1=\sqrt{100a^2}$

$1=10a$

$a=1/10$

$\left(\frac{3}{5}t+1,\ \frac{4}{5}t\right)$

(b) $s'(t) = 3 = \sqrt{100a^2}$

$3 = 10a$

$3/10 = a$

$(9/5t + 1, 12/5t)$

(c) $s'(t) = t = \sqrt{100a^2}$

$t = 10a$

$t/10 = a$

$(\frac{3}{5}t^2 + 1, 4t^2/5)$

(d) $s'(t) = t^2 = \sqrt{100a^2}$

$t^2 = 10a$

$t^2/10 = a$

$(3/5t^3 + 1, 4t^3/5)$.

Arc Length

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Speed

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

Tangent Line

need pt, and a slope