Math 114: Worksheet 28

Arc Length, Speed, Surface Area, + Polar Coordinates.

1. Consider the curve parametrized by $C(t) = (t^4, t^6)$.

(a) Find a cartesian equation for this curve.

Let $x^{1/4} = t$; $y = t^6$

$$y = (x^{1/4})^6$$

$$y = x^{3/2}$$

(b) Find the arc length for the curve for $0 \le t \le 1$. which part of the curve given in part (a) does this compute?

$$\int_0^1 \sqrt{(4t^3)^2 + (6t^5)^2}\, dt$$

$$= \int_0^1 \sqrt{16t^6 + 36t^{10}}\, dt$$

$$= \int_0^1 \sqrt{4t^6(4+9t^4)}\, dt$$

$$= \int_0^1 2t^3\sqrt{4+9t^4}\, dt$$

let $u = 4+9t^4$

$du = 36t^3\, dt$

$\frac{1}{18} du = 2t^3\, dt$

$$= \frac{1}{18}\int_4^{13} \sqrt{u}\, du$$

$$= \frac{1}{18}\left(\frac{2}{3}\right)u^{3/2}\Big|_4^{13} = \frac{1}{27}u^{3/2}\Big|_4^{13} = \frac{1}{27}(13)^{3/2} - \frac{8}{27} \approx \underline{1.4397}$$

computes from the point $(0,0)$ to $(1,1)$.

(c). Find the arclength for this curve for $-1 \le t \le 1$. Which part of the curve given in part (a) does this compute

we get double the amount in part (b), counts when x goes - the other direction. 2.879

2. A "logarithmic spiral" is parametrized by $c(t) = (e^t \cos(t), e^t \sin(t))$.

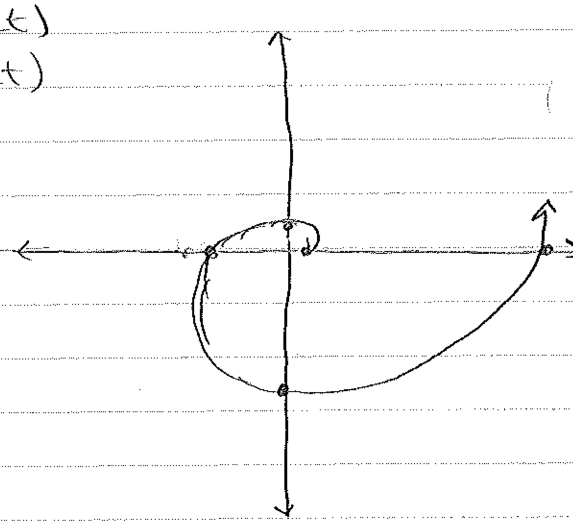
(a) Find the slope of the tangent lines, and use this to sketch the curve, for $0 \le t \le 2\pi$.

$y'(t) = e^t \sin(t) + e^t \cos(t)$

$x'(t) = e^t \cos(t) - e^t \sin(t)$

$$\text{slope} = \frac{\sin(t) + \cos(t)}{\cos(t) - \sin(t)}$$

t	(x,y)
0	$(1,0)$ $m = 1$
$\frac{\pi}{2}$	$(0, e^{\pi/2}) \approx 4.81$ $m = -1$
π	$(-e^{\pi}, 0) \approx -23.14$ $m = 1$
$3\pi/2$	$(0, -e^{3\pi/2}) \approx -111.3177$ $m = -1$
2π	$(e^{2\pi}, 0) \approx 535.49$ $m = 1$



(b) Find the speed $s'(t)$.

$$\text{speed} = \frac{ds}{dt} = \sqrt{(e^t(\cos(t) - \sin(t)))^2 + (e^t(\sin(t) + \cos(t)))^2}$$

$$= \sqrt{e^{2t}(\cos^2(t) - 2\sin(t)\cos(t) + \sin^2(t)) + e^{2t}(\sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t))}$$

$$= \sqrt{e^{2t} + e^{2t}}$$

$$= \sqrt{2}\, e^t$$

(c) Find the length of the curve, again for $0 \le t \le 2\pi$.

$$\int_0^{2\pi} \sqrt{2}e^t \, dt = \sqrt{2}e^t \Big|_0^{2\pi} = \sqrt{2}e^{2\pi} - \sqrt{2}.$$

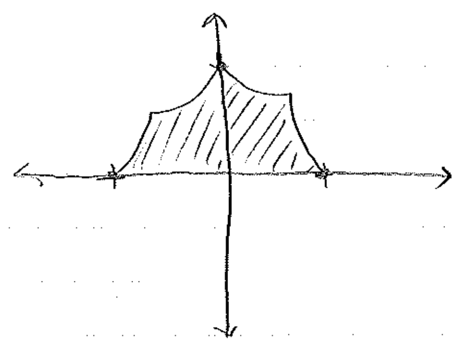
(d) What does the curve look like, for $-2\pi \le t \le 0$?

A much smaller spiral converging on the origin.

3. The curve parametrized by $c(t) = (\cos^3(t), \sin^3(t))$ is known as the "astroid".

(a) Sketch this curve from $0 \le t \le \pi$.

t	(x, y)	
0	$(1, 0)$	$m = 0$
$\pi/4$	$(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$	$m = 1$
$\frac{\pi}{2}$	$(0, 1)$	$m =$ undef.
$3\pi/4$	$(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$	$m = -1$
π	$(-1, 0)$	$m = 0$



$$\frac{3\sin^2(t)\cos(t)}{-3\cos^2(t)\sin(t)} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$$

(b) Find the length of this curve.

$$\int_0^{\pi} \sqrt{(9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} \, dt$$

$$= \int_0^{\pi} \sqrt{9\cos^2(t)\sin^2(t)\left((\cos^2(t) + \sin^2(t)\right)} \, dt$$

$$= 2\int_0^{\frac{\pi}{2}} 3\cos(t)\sin(t) \, dt \quad \text{let } u = \sin(t),$$

$$du = \cos(t)\,dt$$

$6\int_0^1 u\,du = 6(\frac{1}{2}u^2)\Big|_0^1 = 3u^2\Big|_0^1 = 3$

(c) Find the area of the surface obtained by revolving the astroid around the x-axis.

$2\pi\int_0^{\pi/2} \sin^3(t)\,3\cos(t)\sin(t)\,dt$

$= 6\pi\int_0^{\pi/2} \sin^4(t)\cos(t)\,dt$ let $u=\sin(t)$

$du=\cos(t)dt$

$= 6\pi\int_0^1 u^4\,du$

$= \frac{6}{5}\pi u^5\Big|_0^1$

$= \frac{6}{5}\pi$.

4. Convert from rectangular to polar coordinates: $(r,\emptyset)$ $\frac{1}{2}$ $\frac{\sqrt{3}}{2}$

(a) $(1,\sqrt{3})$

$r^2=1+3$ $\tan\emptyset=\sqrt{3}$

$r=2$ $\emptyset=\tan^{-1}(\sqrt{3})$

$\emptyset=\pi/3$

$(2,\pi/3)$

(b) $(-1,0)$ $(1,\pi)$

$r^2=1+0$ $\tan\emptyset=0$

$r=1$ $\emptyset=\tan^{-1}(0)$

$=0,\pi$, take π b/c 2nd QUAD

(c) $(2,-2)$

$r^2 = 4+4$ $\quad$ $\tan\theta = -1$

$r = \sqrt{8} = 2\sqrt{2}$ $\quad$ $\theta = \tan^{-1}(-1)$

$\theta = 3\pi/4$ or $7\pi/4$ / b/c we're in the 4th QUADRANT.

$(2\sqrt{2}, 7\pi/4)$

5. Convert from polar to rectangular:

(a) $(2, \pi/6)$

$x = 2\cos(\pi/6)$ $\quad$ $y = 2\sin(\pi/6)$

$= 2(\sqrt{3}/2)$ $\quad$ $= 2(1/2)$

$= \sqrt{3}$ $\quad$ $= 1$

$(\sqrt{3}, 1)$

(b) $(-1, \pi/2)$

$x = -1\cos(\pi/2)$ $\quad$ $y = -1\sin(\pi/2)$

$= -1(0)$ $\quad$ $= -1(1)$

$= 0$ $\quad$ $= -1$

$(0, -1)$

(c) $(1, -\pi/4)$

$x = 1\cos(7\pi/4)$ $\quad$ $y = 1\sin(7\pi/4)$

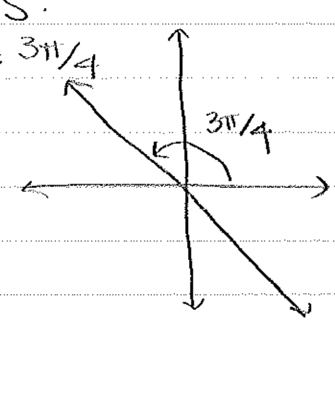
$= 1(\sqrt{2}/2)$ $\quad$ $= 1(-\sqrt{2}/2)$

$= \sqrt{2}/2$ $\quad$ $= -\sqrt{2}/2$

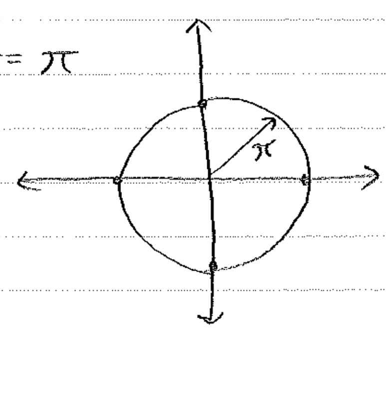
$(\sqrt{2}/2, -\sqrt{2}/2)$.

6. Sketch the graph of the polar curves:

(a) $\theta = 3\pi/4$



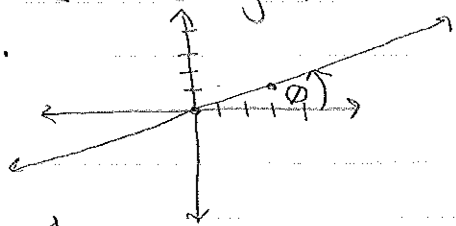
(b) $r = \pi$



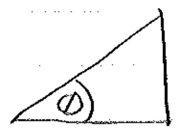
7. Find the equation in polar coordinates of the line through the origin with slope 1/3.

Equations of lines take the form

$\emptyset =$ ________



$\tan^{-1}(1/3) = \emptyset$



8. Find the polar equation for:

(a) $x^2+y^2=9$

$(r\cos\emptyset)^2+(r\sin\emptyset)^2=9$

$r^2(\cos^2\emptyset+\sin^2\emptyset)=9$

$r^2=9$

$r=3$ (circle w/ radius 3)

(b) $x=4$

$r\cos\emptyset=4$

(c) $y=4$

$r\sin\emptyset=4$

9. Convert the equation of the circle $r=2\sin\emptyset$ to rectangular coordinates and find the center and radius of the circle.

$r=2\sin\emptyset$

$r^2=2r\sin\emptyset$

$x^2+y^2=2y$

$x^2+y^2-2y^{+1}=0^{+1}$ (complete the square)

$x^2+(y-1)^2=1$ center: $(0,1)$

radius: 1

10. Given the circle represented by $x^2+(y-2)^2=4$.

(a) Find the polar representation for this equation.

$$(r\cos\theta)^2+(r\sin\theta-2)^2=4$$

$$r^2\cos^2\theta+r^2\sin^2\theta-4r\sin\theta+4=4$$

$$r^2(\cos^2\theta+\sin^2\theta)-4r\sin\theta=0$$

$$r^2-4r\sin\theta=0$$

$$r^2=4r\sin\theta$$

$$\boxed{r=4\sin\theta}$$

(b) Calculate the area enclosed by $0\leq\theta\leq\pi/4$

$$\int_0^{\pi/4}4\sin\theta\,d\theta$$

$$=-4\cos\theta\Big|_0^{\pi/4}=-4\cos(\pi/4)+4\cos(0)$$

$$=-4(\sqrt{2}/2)+4(1)$$

$$=-2\sqrt{2}+4$$

(c) Sketch the area calculated.