Math 114: Worksheet # 29

1. Integration: Compute each of the following (unless the integral is divergent).

(a) $\int \frac{\sin(\ln(t))}{t}\,dt$ let $u=\ln(t)$

$du=\frac{1}{t}dt$

$=\int \sin(u)\,du$

$=-\cos(u)+C$

$=-\cos(\ln(t))+C$

(b) $\int e^x\sin(x)\,dx$ integration by parts

let $u=\sin x$ $dv=e^x dx$

$du=\cos(x)$ $v=e^x$

$I=e^x\sin(x)-\int e^x\cos(x)\,dx$ integration by

$=e^x\sin(x)-(e^x\cos(x)+\int e^x\sin x\,dx)$ parts again

$m=\cos(x)$ $dn=e^x dx$

$2I=e^x\sin(x)-e^x\cos(x)$ $dm=-\sin(x)$ $n=e^x$

$I=\frac{1}{2}(e^x\sin(x)-e^x\cos(x))+C$

(c) $\int_0^1 \frac{x-1}{\sqrt{x}}\,dx=\int_0^1\sqrt{x}\,dx-\int_0^1\frac{1}{\sqrt{x}}\,dx$

$=\frac{2}{3}x^{3/2}\Big|_0^1-\lim_{R\to 0^+}\int_R^1 x^{-1/2}\,dx$

$-\lim_{R\to 0^+} 2x^{1/2}\Big|_R^1$

$-[2(1)^{1/2}-2(R)^{1/2}]$

-2

$=\frac{2}{3}x^{3/2}\Big|_0^1-2$

$=\frac{2}{3}-\frac{6}{3}$

$=-4/3$

(d) $\int_0^1 \frac{x-1}{x^2+3x+2}\,dx \qquad \frac{x-1}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$

$x-1=(x+1)A+(x+2)B$

$x=-1$:

$-2=B$

$x=-2$:

$-3=-1A$

$3=A$

$= \int_0^1 \frac{3}{(x+2)}\,dx$

$+ \int_0^1 \frac{-2}{(x+1)}\,dx$

$= 3\int_0^1 \frac{1}{(x+2)}\,dx - 2\int_0^1 \frac{1}{(x+1)}\,dx$

$= 3\ln|x+2|\Big|_0^1 - 2\ln|x+1|\Big|_0^1$

$= 3\ln|3| - 3\ln|2| - (2\ln|2| - \overset{0}{\cancel{2\ln|1|}})$

$= 3\ln|3| - 3\ln|2| - 2\ln|2|$

$= 3\ln|3| - 5\ln|2|$

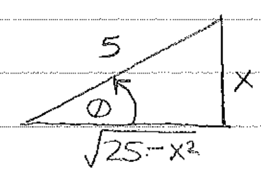
$\approx -.169899$

(e) $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx \qquad$ let $x=5\sin\theta \qquad x^2=25\sin^2\theta$

$dx = 5\cos\theta\, d\theta$

$= \int \frac{1}{25\sin^2\theta\sqrt{25-25\sin^2\theta}}\, 5\cos\theta\, d\theta$

$= \int \frac{1}{25\sin^2\theta}\, d\theta$



$= \int \frac{1}{25}\csc^2\theta\, d\theta$

$= -\frac{1}{25}\cot\theta + C$

$= -\frac{1}{25}\left(\frac{\sqrt{25-x^2}}{x}\right) + C.$

(f) $\int_{-\infty}^{0} xe^x\,dx$

Now use integration by parts

$= \lim\limits_{R\to-\infty} \int_R^0 xe^x\,dx$ $\quad u = x \quad dv = e^x\,dx$

$\quad du = dx \quad v = e^x$

$= \lim\limits_{R\to-\infty} xe^x - \int_R^0 e^x\,dx$

$= \lim\limits_{R\to-\infty} xe^x - e^x \Big|_R^0$

$= \lim\limits_{R\to-\infty} \underbrace{(0e^0 - e^0)}_{-1} - (Re^R - e^R)$

$= -1 - \lim\limits_{R\to\infty} \dfrac{R-1}{e^{-R}}$ L'hopitals Rule

$\underbrace{-\lim\limits_{R\to-\infty} \dfrac{1}{-e^{-R}}}$

$= 0$

$= -1 - 0 = -1$

3. Areas, Volumes, and Lengths: set up (but do not evaluate) integrals for the following geometric quantities.

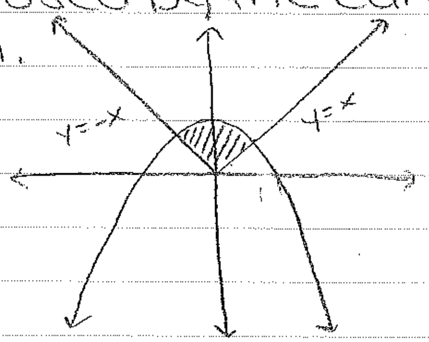
(a) The area enclosed by the curves $y = 1 - 2x^2$, $y = |x|$.

$1 - 2x^2 = x$

$0 = 2x^2 + x - 1$

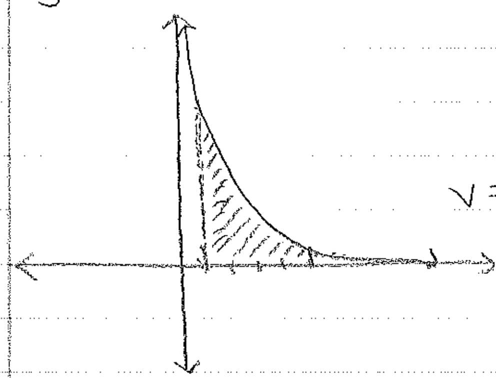
$= (2x - 1)(x + 1)$

$x = 1/2$ or $x = -1$



$2\int_0^{\frac{1}{2}} (1 - 2x^2) - x\,dx$

(b) the volume obtained by rotating the region bounded by the curves $y = 1/x$, $x = 1$, and $x = 5$ about the x-axis.



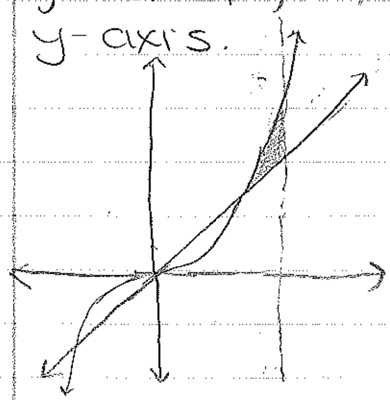
DISK METHOD

$$V = \pi \int_1^5 \left(\frac{1}{x}\right)^2 dx$$

(c) the volume obtained by rotating the region bounded by the curves $y = 1/x$, $x = 1$, and $x = 5$ about the y-axis.

$$V = 2\pi \int_1^5 x\left(\frac{1}{x}\right) dx$$

(d) the volume obtained by rotating the region bounded by the curves $y = \tan x$, $y = x$, and $x = \pi/3$ about the y-axis.



cylindrical shells

$$V = 2\pi \int_0^{\pi/3} x(\tan x - x)$$

lower bound

$x = \tan(x)$

(e) the length of the curve $y=x^2$ from $x=a$ to $x=b$.

$$S=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_a^b \sqrt{1+4x^2}\,dx$$

(f) the length of the parametric curve $x=3t^2$, $y=2t^3$, $0\leq t\leq 2$

$$S=\int_0^2 \sqrt{(6t)^2+(6t^2)^2}\,dt$$

$$=\int_0^2 \sqrt{36t^2+36t^4}\,dt$$

4. Parametric Equations

(a) Consider the curve $x(t)=3t^2+t$ and $y(t)=2t$.

(i) Eliminate the parameter, t, to find a Cartesian equation for the curve.

$x=3t^2+t$, $y=2t \rightarrow \frac{1}{2}y=t$

$x=3(\frac{1}{2}y)^2+(\frac{1}{2}y)$

$x=\frac{3}{4}y^2+\frac{1}{2}y$

(ii) Find the tangent line to this curve at the point $(x,y)=(14,4)$

$m=\frac{y'(t)}{x'(t)}=\frac{2}{6t+1}$ $\qquad y=4=2t$, $t=2$

$m(2)=\frac{2}{13}$

$4=(\frac{2}{13})(14)+b$

$\frac{52}{13}=\frac{28}{13}+b$

$24/13=b$

$y=2/13\,x+24/13$.

5. Differential Equations

(a) Solve $\frac{dL}{dt} = kL^2 \ln t$, $L(1) = -1$.

$$\int \frac{dL}{L^2} = \int k \ln t \, dt \qquad \text{let } u = \ln(t) \quad dv = k\,dt$$
$$du = \frac{1}{t}\,dt \quad v = kt$$

$$-\frac{1}{L} = \ln(t)kt - \int k\,dt$$

$$-\frac{1}{L} = \ln(t)kt - kt + C \qquad -1 = \frac{1}{k + C}$$

$$\frac{1}{L} = kt - \ln(t)kt + C \qquad -k - C = 1 \rightarrow C = -k - 1$$

$$L = \frac{1}{kt - \ln(t)kt + C} \qquad L = \frac{1}{kt - \ln(t)kt - k - 1}$$

(c) Find the solution of $y' - y = e^{2x}$, $y(0) = 1$.

$$\alpha(x) = e^{-\int dx}$$
$$= e^{-x}$$

$$y = \frac{1}{\alpha(x)} \int \alpha(x) e^{2x}\,dx \qquad y(0) = 1$$
$$1 = e^{2(0)} + Ce^{0}$$

$$= \frac{1}{e^{-x}} \int e^{-x} e^{2x}\,dx \qquad 1 = 1 + C$$
$$0 = C$$

$$= e^{x} \int e^{x}\,dx$$
$$= e^{x}(e^{x} + C) \qquad y = e^{2x}$$
$$= e^{2x} + Ce^{x}$$

6. Sequences and Series

(a) Does the sequence $\left\{\frac{2+n^3}{4+5n^3}\right\}$ converge? If so, what is its limit?

Yes, the sequence converges to $1/5$ because $\lim_{n\to\infty} \frac{2+n^3}{4+5n^3} = \frac{1}{5}$.

(b) True or False: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

FALSE! Incorrect application of divergence test.

(c) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ converge conditionally, converge absolutely, or diverge? Explain.

Leibniz Alternating Series Test: consider the series $\{\frac{1}{n^{1/3}}\}$, positive ✓

decreasing (derivative is always negative) ✓

$\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$ ✓

Thus the series converges conditionally.

Consider $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^{1/3}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$, which diverges by the p-test. $(p = 1/3 \leq 1)$.

(d) $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ Consider $S_1 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_2 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

(Telescoping Series) $S_3 = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5}$

$S_N = 1 - \frac{1}{N+2}$

$\lim_{N\to\infty} S_N = 1 - \frac{1}{N+2} = 1.$

(e) Test the following series for convergence.

(i) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ use the integral test.

Note that $n\sqrt{\ln(n)}$ is positive and $\frac{1}{n\sqrt{\ln(n)}}$ is decreasing: $\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx$ let $u = \ln(x)$ $du = \frac{1}{x}dx$

$\lim_{R\to\infty} \int_2^R u^{-1/2}\,du$

$= \lim_{R\to\infty} 2u^{1/2}\Big|_2^R = \lim_{R\to\infty} (2R^{1/2} - 2\sqrt{2}) = \infty$, So the integral diverges and thus the series diverges as well.

(ii) $\sum_{n=1}^{\infty} \frac{n^7}{7^n}$ Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(n+1)^7}{7^{n+1}} \cdot \frac{7^n}{n^7} \right| = \lim_{n\to\infty} \frac{1}{7} \left| \frac{(n+1)^7}{n^7} \right| = \frac{1}{7},$$

so the series converges absolutely.

(iii) $\sum_{n=0}^{\infty} \frac{\cos(n)}{2+2^n} \le \sum_{n=0}^{\infty} \left| \frac{\cos(n)}{2+2^n} \right|$

$$\le \sum_{n=0}^{\infty} \left| \frac{\cos(n)}{2^n} \right|$$

$$\le \sum_{n=0}^{\infty} \left| \frac{1}{2^n} \right|,$$ which is a geometric series, that converges because $r < 1$.

(iv) $\sum_{n=1}^{\infty} \frac{5^n+n^2+n+17}{3n^4+4^n+6}$ Use the limit comparison with $\{5^n/4^n\}$

$$\lim_{n\to\infty} \frac{5^n+n^2+n+17}{3n^4+4^n+6} \cdot \frac{\frac{1}{5^n}}{\frac{1}{4^n}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{n^2}{5^n}+\frac{n}{5^n}+\frac{17}{5^n}}{\frac{3n^4}{4^n}+1+\frac{6}{4^n}} = \frac{1}{1} = 1,$$ so the series diverges

7. Power, Maclaurin, and Taylor Series

(a) Find the Maclaurin Series for $x^2/1+x$.

$$\frac{x^2}{1+x} = x^2\left(\frac{1}{1-(-x)}\right) = x^2 \sum_{n\ge 0}(-x)^n = \sum_{n\ge 0}(-1)^n x^{n+2}$$

(b) Find the Taylor Series for cos(x) about $a = \pi/2$.

$f(\pi/2) = \cos(\pi/2) = 0$

$f'(\pi/2) = -\sin(\pi/2) = -1$

$f''(\pi/2) = -\cos(\pi/2) = 0$

$f'''(\pi/2) = \sin(\pi/2) = 1$

$f''''(\pi/2) = \cos(\pi/2) = 0$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi/2)^{2n+1}$$

8. Use the limit comparison test to determine if $\sum_{n=1}^{\infty}(1-\cos\frac{1}{n})$ converges or diverges.

Compare with $\frac{1}{n^2}$ which converges.

$\lim_{n\to\infty} \frac{(1-\cos\frac{1}{n})}{1/n^2} = \lim_{n\to\infty} n^2 - n^2\cos(\frac{1}{n}) = 0$, thus by the limit comparison test, we have that $\sum_{n=1}^{\infty}(1-\cos(\frac{1}{n}))$ converges.