Math 114 Worksheet 3

Sequences

1. Write the first four terms of the sequences with the following general terms:

(a) $n!/2^n$, If $n=0$: $0, 1/2, 1/2, 3/4, 3/2$

(b) $n/n+1$, If $n=0$: $0, 1/2, 2/3, 3/4, 4/5$

(c) $(-1)^{n+1}$, If $n=0$: $-1, 1, -1, 1, -1$

2. Find a formula for the n^{th} term of the sequence $\{\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \ldots\}$

$a_n = (-1)^{n+1}/n^3$

3. Conceptual Understanding:

(a) What is a sequence?

A sequence is an ordered collection of numbers defined by a function $f(n)$ on a set of integers.

(b) What does it mean to say that a sequence is bounded?

A sequence is called bounded if it is bounded from above and below. A sequence is bounded from above if there is a number M such that $a_n \leq M$ for all n. The number M is called an upper bound.

(c) What does it mean to say that a sequence is defined recursively?

A sequence that is defined recursively has a known first term and then the n^{th} term, a_n, is computed in terms of the preceding term a_{n-1}.

(d) What does it mean to say that a sequence converges?

A sequence converges if $\lim_{n\to\infty} a_n = L$, that is, as n approaches infinity, the terms of the sequence approach a number, L.

4. Let $a_0 = 0$ and $a_1 = 1$. Write out the first five terms of $\{a_n\}$ where a_n is recursively defined as $a_{n+1} = 3a_{n-1} + a_n^2$.

$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ a_4 = 19,\ a_5 = 373$

5. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, produce a counterexample.

Consider $a_n = (-1)^n = 1, -1, 1, -1, \ldots$ does not converge, but is bounded above and below.

6. Show that the sequence with general term $a_n = 3n^2/n^2+2$ is increasing. Find an upper bound. Does $\{a_n\}$ converge?

Consider $f(x) = 3x^2/x^2+2$. Now find

$$f'(x) = \frac{(x^2+2)(6x) - 3x^2(2x)}{(x^2+2)^2} = \frac{6x^3 + 12x - 6x^3}{(x^2+2)^2}$$

$$= \frac{6x^3}{(x^2+2)^2}$$, which is positive. Thus the sequence is increasing.

$M = 3$ is an upper bound on $3n^2/n^2+2$ because $3 = 3x^2/x^2 \geq 3x^2/x^2+2$ for all $x > 0$.

Since a_n is bounded and monotonic,

it follows that $\{a_n\}$ converges.

7. Use appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

(a) $a_n = 1.01^n$, This is a geometric sequence with $r>1$, thus the sequence diverges.

(b) $b_n = \frac{3n^2+n+1}{2n^2-3}$, consider $\lim_{n\to\infty} \frac{3n^2+n+1}{2n^2-3} = 3/2$

(c) $c_n = e^{1-n^2}$, consider $\lim_{n\to\infty} e^{1-n^2} = e^{-\infty} = 0$.