Math 114 Worksheet 4
Summing an Infinite Series

1. Compute the following sums

(a) $\sum_{n=1}^{5} 3n = 3\sum_{n=1}^{5} n = 3(1+2+3+4+5) = 3(15) = 45$

(b) $\sum_{k=3}^{6} (\sin(\frac{\pi}{2}+\pi k) + 2k)$

$= (\sin(\frac{\pi}{2}+3\pi)+6) + (\sin(\frac{\pi}{2}+4\pi)+8)$
$+ (\sin(\frac{\pi}{2}+5\pi)+10) + (\sin(\frac{\pi}{2}+6\pi)+12)$

$= (\sin(\frac{7\pi}{2})+6) + (\sin(\frac{9\pi}{2})+8)$
$+ (\sin(\frac{11\pi}{2})+10) + (\sin(\frac{13\pi}{2})+12)$

$= (-1+6)+(1+8)+(-1+10)+(1+12)$

$= 5+9+9+13$

$= 36$

2. Conceptual understanding:

(a) What is a series? A series is the sum of the terms of a sequence. Finite series have defined first and last terms, whereas infinite series continue indefinitely.

(b) What is the difference between a sequence and a series?

A sequence is a collection of ordered terms, whereas a series is the sum of terms of a sequence.

(c) What does it mean that a series converges?

An infinite series converges to the sum S if its partial sums converge to S, $\lim_{N\to\infty} S_N = S$.

(d) What is the sequence of partial sums?

The partial sums S_N are defined

as the finite sum of the terms up to and including the N^{th} term.

3. Write the following in summation notation:

(a) $\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+\dots$

$=\sum_{n=3}^{\infty}\frac{1}{n^2}$

(b) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$

$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}$

4. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series

$S=\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

$S_3=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)=\frac{1}{2}-\frac{1}{5}=3/10$

$S_4=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)=\frac{1}{2}-\frac{1}{6}=1/3$

$S_5=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)=\frac{1}{2}-\frac{1}{7}=5/14$

The sum S is the limit of the partial sums: $S=\lim_{N\to\infty}S_N=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+2}\right)=\frac{1}{2}$.

5. Use Theorem 3 of 10.2 (Divergence Test) to prove that the following two series diverge:

*Theorem 3 (Divergence Test): If the n^{th} term a_n does not converge to zero, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

(a) $\sum_{n=1}^{\infty}\frac{n}{10n+12}$

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{10n+12}=\lim_{n\to\infty}\frac{1}{10+12/n}=\frac{1}{10}$ (DIVERGES)

(b) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{n}{\sqrt{n^2(1+1/n^2)}}=\lim_{n\to\infty}\frac{n}{n\sqrt{1+1/n^2}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}}=1.$

(DIVERGES)

6. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:

(a) $1+\frac{1}{8}+\frac{1}{8^2}+\ldots$

$\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n$, $r=\frac{1}{8}$, so $|r|<1$, thus $\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n=\frac{1}{1-\frac{1}{8}}=8/7$.

(b) $\sum_{n=0}^{\infty}\left(\frac{\pi}{e}\right)^n$, $r=\frac{\pi}{e}\approx 1.1557$, so $|r|>1$, thus the series diverges.

(c) $5-\frac{5}{4}+\frac{5}{4^2}-\frac{5}{4^3}+\ldots$

$\sum_{n=0}^{\infty}5\left(\frac{-1}{4}\right)^n$, $r=\frac{-1}{4}$, so $|r|<1$, thus

$\sum_{n=0}^{\infty}5\left(\frac{-1}{4}\right)^n=5\left(\frac{1}{1-\left(\frac{-1}{4}\right)}\right)=5\left(\frac{1}{5/4}\right)=5\left(\frac{4}{5}\right)=4.$

(d) $\sum_{n=0}^{\infty}\frac{8+2^n}{5^n}=\sum_{n=0}^{\infty}\frac{8}{5^n}+\sum_{n=0}^{\infty}\frac{2^n}{5^n}=8\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n+\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$

$=8\left(\frac{1}{1-\frac{1}{5}}\right)+\left(\frac{1}{1-2/5}\right)=8\left(\frac{5}{4}\right)+\left(\frac{5}{3}\right)=10+\frac{5}{3}=35/3$

case 1: $r=\frac{1}{5}$, so $|r|<1$

case 2: $r=2/5$, so $|r|<1$.

Geometric series: If $|r|<1$, then $\sum_{n=0}^{\infty}r^n=1+r+r^2\ldots=\frac{1}{1-r}$ and $\sum_{n=M}^{\infty}cr^n=cr^M+cr^{M+1}+\ldots=cr^M/1-r$.

The geometric series diverges if $|r|\geq 1$.

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21, 25, 40, 41 (a,c)

21. $\lim_{n\to\infty}a_n=\cos\left(\frac{1}{n+1}\right)=1\neq 0$.

25. The series diverges.

41. (a) counterexample: $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1$.

(c) counterexample: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.