Math 114 Worksheet 5

Series with Positive Terms

Theorem 2 (Integral Test): Let $a_n = f(n)$, where $f(x)$ is positive, decreasing, and continuous for $x \geq 1$.

(i) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

1. Use the Integral Test to determine if the following series converge or diverge:

(a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$. Clearly $a_n = f(n)$, where $f(x) = \frac{1}{1+x^2}$. Note that $f(x)$ is positive, decreasing, and continuous for $x \geq 1$.

$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \tan^{-1}(x)\Big|_1^R$

$= \lim_{R\to\infty} (\tan^{-1}(R) - \tan^{-1}(1)) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, since the integral converges, we know the series converges.

(b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$. Clearly $a_n = f(n)$, where $f(x) = \frac{x^2}{e^{x^3}}$. Note that $f(x)$ is positive, decreasing, and continuous for $x \geq 1$.

$\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{R\to\infty} \int_1^R x^2 e^{-x^3}\,dx$. let $u = -x^3$

$= \lim_{R\to\infty} \int_1^R -\frac{1}{3} e^u\,du$ $\qquad du = -3x^2\,dx$

$\qquad -\frac{1}{3}du = x^2\,dx$

$= \lim_{R\to\infty} -\frac{1}{3} e^{-x^3}\Big|_1^R = \lim_{R\to\infty} (-\frac{1}{3}e^{-R^3} - (-\frac{1}{3}e^{-1}))$

$= \lim_{R\to\infty} (-\frac{1}{3}e^{-R^3} + \frac{1}{3e}) = (0 + \frac{1}{3e}) = \frac{1}{3e}$. Since the integral converges, we know the series diverges.

(c) $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$. Clearly $a_n = f(n)$, where $f(x) = x(x^2+2)^{-3/2}$. Note that $f(x)$ is positive, decreasing, and continuous for $x \geq 2$.

$\int_2^{\infty} x(x^2+2)^{-3/2}\,dx = \lim_{R\to\infty}\int_2^R \frac{x}{(x^2+2)^{3/2}}\,dx$. Let $u = x^2+2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \lim_{R\to\infty} \frac{1}{2}\int_2^R u^{-3/2}\,du$

$= \lim_{R\to\infty} \frac{1}{2}\left(-2u^{-1/2}\right)\Big|_2^R = \lim_{R\to\infty} -(x^2+2)^{-1/2}\Big|_2^R$

$= \lim_{R\to\infty}\left(\frac{-1}{\sqrt{R^2+2}} - \left(\frac{-1}{\sqrt{6}}\right)\right) = \left(0+\frac{1}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$. Since the integral converges, then we know that the series converges.

2. Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise by Integral Test.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$. Clearly $a_n = f(n)$, where $f(x) = \frac{1}{x^p}$. Note that $f(x)$ is positive, decreasing, and continuous when $x \geq 1$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p \leq 1 \end{cases}$$ by theorem 1 in section 7.6. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

3. Use the Comparison Test (or Limit Comparison Test) to determine whether the infinite series is convergent or divergent.

Theorem 4 (comparison test): assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Theorem 5 (Limit Comparison Test): let $\{a_n\}$ and $\{b_n\}$ be positive sequences. Assume that the following limit exists: $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

* If $L>0$, then $\sum a_n$ converges iff $\sum b_n$ converges.
* If $L=\infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
* If $L=0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges so the initial series converges.

(b) $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$. Apply the Limit Comparison Test with $a_n = \frac{2}{\sqrt{n^2+2}}$ and $b_n = \frac{1}{n}$. Then $L = \lim \frac{a_n}{b_n} = \lim_{n\to\infty} 2n/\sqrt{n^2+2} = \lim_{n\to\infty} 2/\sqrt{1+2/n^2} = 2$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $L>0$, the series $\sum_{n=1}^{\infty} 2/\sqrt{n^2+2}$ also diverges.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{2+5^n} \leq \sum_{n=1}^{\infty} \frac{2^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a geometric series, where $r = 2/5$, so since $|r|<1$, we know the series converges.

(d) $\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$. $\frac{4^n+2}{3^n+1} > \frac{4^n}{3^n+3^n} = \frac{1}{2}\left(\frac{4}{3}\right)^n$, which is a geometric series, where $r = 4/3$, so since $|r|>1$, we know the series diverges.

(e) $\sum_{n=1}^{\infty} \frac{n!}{n^4} = \frac{(n-1)!}{n^3} > \frac{(n-1)(n-2)(n-3)}{n^3} = \frac{n^3-6n^2+11n-6}{n^3}$

$\sum_{n=1}^{\infty} \frac{n^3-6n^2+11n-6}{n^3}$, which diverges by the n^{th} term test. So the initial series must diverge.

(f) $\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!} = \sum_{n=0}^{\infty} \frac{n}{n+1} \cdot \frac{n}{n} \cdot \frac{1}{(n-1)!} \leq \sum_{n=0}^{\infty} \frac{1}{(n-1)!} \leq \sum_{n=0}^{\infty} \frac{1}{n^2}$, which converges so the entire series converges