Math 114: Worksheet 7

Ratio and Root Test and Power Series

1. (a) State the Ratio Test

Assume that the following limit exists: $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(i) If $\rho < 1$, then $\sum a_n$ converges absolutely.

(ii) If $\rho < 1$, then $\sum a_n$ diverges.

(iii) If $\rho = 1$, the test is inconclusive.

(b) State the Root Test.

Assume that the following limit exists: $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

(i) If $L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $L > 1$, then $\sum a_n$ diverges.

(iii) If $L = 1$, the test is inconclusive.

2. Determine whether the series is convergent or divergent.

(a) $\sum_{n=0}^{\infty} \left(3n^3+2n/4n^3+1\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left|3n^3+2n/4n^3+1\right|^n} = \lim_{n\to\infty} \left|3n^3+2n/4n^3+1\right| = 3/4$.

so it converges absolutely.

(b) $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$

$\lim_{n\to\infty} \left|\frac{2^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{2^n n^2}\right| = \left|\frac{2(n+\cancel{1})^2}{(\cancel{n+1})n^2}\right| = \left|\frac{2n+2}{n^2}\right|$

$\lim_{n\to\infty} \left|\frac{2n+2}{n^2}\right| = 0$, so it converges absolutely.

(c) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

$\lim_{n\to\infty} \left|\frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}\right| = \lim_{n\to\infty} \left|\frac{e}{n+1}\right| = 0$, so it converges absolutely.

3. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.

(a) To apply the Ratio Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If this limit is less than 1 then the series converges absolutely.

TRUE

(b) To apply the Root Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is larger than 1 the series diverges.

FALSE (corrected statement above)

4. Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$.

Every power series has a radius of convergence R, which is either nonnegative or infinity, converges absolutely when $|x-c|<R$

5. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{4n} \right| (x-3)$$

WTS $-1 < \frac{1}{4}(x-3) < 1$

$-4 < x-3 < 4$

$-1 < x < 7$, so the radius of convergence is 4.

interval of convergence: we know $(-1,7)$ converges, but what about the endpoints: -1 and 7.

$\underline{x=-1}$ $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-1-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=0}^{\infty}(-1)^{2n} n$, which diverges.

$\underline{x=7}$ $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(7-3)^n = \sum_{n=0}^{\infty}(-1)^n n$, which diverges.

So the interval of convergence is $(-1,7)$.

6. Find the radius and interval of convergence for $4\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n = \sum_{n=0}^{\infty} \frac{2^{n+2}}{n}(4x-8)^n$

$$\lim_{n\to\infty}\left|\frac{2^{n+3}(4x-8)^{n+1}}{n+1}\cdot\frac{n}{2^{n+2}(4x-8)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2n}{n+1}\right|(4x-8) \quad \text{w is} \quad -1<8x-16<1$$

$$15<8x<17$$

$$15/8<x<17/8$$

So the radius of convergence is $1/8$

interval of convergence: we know $(15/8, 17/8)$ converges, but what about the endpoints.

$\underline{x=\frac{15}{8}}$ $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{4(-1)^n}{n}$, which converges by the alternating series test.

$\underline{x=\frac{17}{8}}$ $\sum_{n=0}^{\infty} \frac{2^{n+2}}{n}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{4}{n}$, which diverges by the comparison test with the harmonic series.

So the interval of convergence is $\left[\frac{15}{8}, \frac{17}{8}\right)$.

7. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \left(\frac{x^2}{-3}\right)^n$

$\lim_{n\to\infty} \left| \frac{(x^2)^{n+1}}{(-3)^{n+1}} \cdot \frac{-3^n}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{-3} \right|$, we want

$x^2/3 < |1| \Rightarrow x^2 < 3 \Rightarrow \sqrt{3} < x < \sqrt{3}$.

Radius of Convergence is $\sqrt{3}$.

Now we need to check $\pm\sqrt{3}$ for convergence.

$\underline{x = \sqrt{3}}$ $\sum_{n=0}^{\infty} \left(\frac{\sqrt{3}^2}{-3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{-3}\right)^n = \sum_{n=0}^{\infty} (-1)^n$, diverges

$\underline{x = -\sqrt{3}}$ $\sum_{n=0}^{\infty} \left(\frac{(-\sqrt{3})^2}{-3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{3}{-3}\right)^n = \sum_{n=0}^{\infty} (-1)^n$, diverges

so the interval of convergence is $(-\sqrt{3}, \sqrt{3})$.

8. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} n!(x-2)^n$

$\lim_{n\to\infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!(x-2)^n} \right| = |n+1||(x-2)| = \infty$,

so the series diverges for all x and the radius of convergence is 0 and the interval of convergence is 0.