Math 114 Worksheet 8

Review for Exam 1

1. Integration by Parts:

(a) $\int x^2\cos(x)\,dx$

Let $u = x^2$ $\quad v' = \cos(x)\,dx$

$\quad du = 2x\,dx \quad v = \sin(x)$

$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$

Let $u = x$, $v' = \sin(x)$

$du = dx \quad v = -\cos(x)$

$-2(-x\cos(x) + \int \cos(x)\,dx)$

$= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C.$

(b) $\int 2x\arctan(x)\,dx$

Let $u = \arctan(x)$ $\quad v' = 2x\,dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = x^2$

$\int 2x\arctan(x)\,dx = x^2\arctan(x) - \int \frac{x^2}{1+x^2}\,dx$

$= x^2\arctan(x) - \int \frac{x^2+1-1}{1+x^2}\,dx$

$= x^2\arctan(x) - (x - \arctan(x))$

$= x^2\arctan(x) - x + \arctan(x) + C.$

2. Improper Integrals:

(a) Evaluate $\int_{-1}^{\infty} e^{-x}\,dx$

$\lim_{R\to\infty} \int_{-1}^{R} e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_{-1}^{R}$

$= \lim_{R\to\infty} -e^{-R} - (-e) = 0 - (-e) = e$

(b) Evaluate $\int_{1}^{2} \frac{dx}{x\ln(x)}$

$\lim_{R\to 1^+} \int_{R}^{2} \frac{1}{x\ln(x)}\,dx = \lim_{R\to 1^+} \int_{u(R)}^{u(2)} \frac{1}{u}\,du = \lim_{R\to 1^+} \ln(u)\Big|_{u(R)}^{u(2)}$

let $u = \ln(x)$ $\qquad = \lim_{R\to 1^+} \ln(\ln(x))\Big|_{R}^{2}$

$du = \frac{1}{x}\,dx$

$= \lim_{R\to 1^+} \underbrace{\ln(\ln(2))}_{\text{number}} - \overbrace{\ln(\ln(1^+))}^{\text{DNE}}$

$\text{number} - (-\infty) = \infty.$

(c) Evaluate $\int_3^6 \frac{x}{\sqrt{x-3}}\,dx$

Let $u = x-3$ $\quad \lim_{R\to 3^+} \int_R^6 \frac{u+3}{\sqrt{u}}\,du$

$x = u+3$

$du = dx$ $\quad = \lim_{R\to 3^+} \int_R^6 u^{1/2} + 3u^{-1/2}\,du$

$= \lim_{R\to 3^+} \frac{2}{3}(x-3)^{3/2} + 6(x-3)^{1/2} \Big|_R^6$

$= \lim_{R\to 3^+} \left(\frac{2}{3}(3)^{3/2} + 6(3)^{1/2}\right) - (0+0) = \frac{2}{3}(3\sqrt{3}) + 6\sqrt{3}$

$= 8\sqrt{3}$

3. Determine the limit of the sequence or state that the sequence diverges.

(a) $a_n = \sqrt{n+3} - \sqrt{n}$ $\left(\text{conjugate}: \frac{\sqrt{n+3}+\sqrt{n}}{\sqrt{n+3}+\sqrt{n}}\right)$

$a_n = \frac{(n+3) - n}{(\sqrt{n+3}+\sqrt{n})}$ $\quad \lim_{n\to\infty} 3/(\sqrt{n+3}+\sqrt{n}) = 0$ ✓

(b) $b_n = \frac{\cos(n)}{n}$

$\lim_{n\to\infty} \frac{-1}{n} = 0 \leq \lim_{n\to\infty} \frac{\cos(n)}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$ ✓

(c) $c_n = \frac{e^n + (-3)^n}{5^n}$ $\quad \lim_{n\to\infty}\left(\frac{e}{5}\right)^n + \lim_{n\to\infty}\left(\frac{-3}{5}\right)^n = 0$

$\cancel{① \; 5/e5 \; + \; ② \; \frac{5}{2}}$

(d) $a_n = n^{1/n}$

$\lim_{n\to\infty} e^{\frac{\ln(n)}{n}}, \; \lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

$\lim_{n\to\infty} e^{\frac{\ln(n)}{n}} = e^0 = 1$ ✓

4. Determine whether or not $\sum_{n=2}^{\infty}(1-\sqrt{1-\frac{1}{n^2}})$ converges.

$$\sum_{n=2}^{\infty}(1-\sqrt{1-\tfrac{1}{n^2}}) = \sum_{n=2}^{\infty}\frac{1-(1-\frac{1}{n^2})}{1+\sqrt{1-1/n^2}} = \sum_{n=2}^{\infty}\frac{1}{n^2+n^2\sqrt{1-1/n^2}}$$

converges by the p-test.

5. Evaluate $\sum_{n=3}^{\infty}\frac{1}{n(n+3)}$.

$$\frac{1}{3}\sum_{n=3}^{\infty}\frac{1}{n}-\frac{1}{n+3} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n+2}-\frac{1}{n+5}$$

$$S_N = \frac{1}{6} - \frac{1}{3N+15}, \lim_{N\to\infty} S_N = \frac{1}{6}$$

6. Find a value of N such that S_N approximates the series with an error of at most 10^{-5} where

$$S = \sum_{n=1}^{\infty} (-1)^{n+1}/n(n+2)(n+3)$$

$$\frac{1}{(n+1)(n+3)(n+4)} < \frac{1.1}{10^5} \Rightarrow 10^5 < (n+1)(n+3)(n+4).$$

N = 44 works.

$|S_N - S| < a_{n+1}$

7. Determine convergence or divergence.

(a) $\sum_{n=0}^{\infty}(\frac{1}{5})^n$, converges, geometric series with $r=\frac{1}{5}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+1}}$, converges conditionally by the Leibniz alternating series test

(c) $\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$, integral test with $f(x)=\frac{x}{e^{x^2}}$, +, decr. cont.

$$\lim_{R\to\infty}\int_1^R \frac{x}{e^{x^2}}dx = \lim_{R\to\infty}\int_1^R \frac{1}{2}e^{-u}du = \lim -\frac{1}{2}e^{-u}\Big|^R$$

let $u = x^2$,

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \lim_{R\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^{R}$$

$$= \lim_{R\to\infty} -\frac{1}{2}e^{-R} - (-\frac{1}{2}e^{-1})$$

$$= \frac{1}{2e}.$$

(d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$, Ratio Test

$\lim_{n\to\infty}\left|\frac{e^{n+1}}{(n+1)!}\cdot\frac{n!}{e^n}\right| = \lim_{n\to\infty}\left|\frac{e}{n+1}\right| = 0$, so the series converges absolutely

(e) $\sum_{n=1}^{\infty}\left(\frac{e}{n}\right)^n$, Root Test

$\lim_{n\to\infty}\left|\frac{e}{n}\right| = 0$, so the series converges absolutely

8. Determine the radius of convergence of the following series:

(a) $\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$

$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n x^n}\right| = \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| = 0$, so the radius of convergence is ∞.

(b) $\sum_{n=2}^{\infty}\frac{(2x-3)^n}{n\ln(n)}$

$\lim_{n\to\infty}\left|\frac{(2x-3)^{n+1}}{(n+1)\ln(n+1)}\cdot\frac{n\ln(n)}{(2x-3)^n}\right| = \left|\overset{\to 1}{\frac{n\ln(n)}{(n+1)\ln(n+1)}}\right|\,|2x-3|$

$|2x-3|<1 \Rightarrow |x-\frac{3}{2}|<\frac{1}{2}$. Check 1, 2

So the radius of convergence is $R=\frac{1}{2}$.

$[1, 2)$.