MA 114 Worksheet # 1: Introduction to Sequences

1. Conceptual Understanding:

   (a) Give the precise definition of a *sequence*.

   (b) Review: What does it mean to say that $\lim_{x\to a} f(x) = L$ when $a = \infty$? Why does this differ from $\lim_{n\to\infty} f(n) = L$?

   (c) Say what it means for a sequence to converge. Provide an intuitive explanation of this concept.

   (d) Sequences can diverge in different ways. Describe two distinct ways that a sequence can diverge.

   (e) Give two examples of sequences which converge to 0 and two examples of sequences which converges to a given number $L \neq 0$.

2. Write out the first four terms of the following sequences. Do the sequences converge? If so, what is the limit?

   (a) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{3}{n}$

   (b) $\{x_k\}_{k=1}^{\infty}$ where $x_k = 2^{-k} + 2$

   (c) $\{b_n\}_{n=1}^{\infty}$ where $b_n = \dfrac{n}{n+1}$

   (d) $\{s_j\}_{j=1}^{\infty}$ where $s_j = \dfrac{(-1)^j}{j^2}$

   (e) $\{F_n\}_{n=0}^{\infty}$ where $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$

3. Find a formula for the general term of the following sequences, assuming that the natural pattern of the first few terms continues.

   (a) $\{1, 1/2, 1/4, 1/8, 1/16, ...\}$

   (b) $\{0, 1, 0, 1, 0, 1, ...\}$

   (c) $\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{9}, -\frac{5}{8}, \frac{6}{27}, ...\}$

4. The limit laws for sequences are very similar to the limit laws for functions. Suppose you have sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} a_n = 15$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 1$. Use the limit laws of sequences to answer the following questions. Carefully justify each step.

   (a) Does the sequence $\left\{\dfrac{a_n \cdot c_n}{b_n + 1}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

   (b) Does the sequence $\left\{\dfrac{a_n + 3 \cdot c_n}{2 \cdot b_n}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

5. In Calculus I, you learned many techniques for taking limits of functions. The following theorem suggests how you might apply what you already know about the limits of functions to the study of sequences:

   Theorem. Given a sequence $\{a_n\}$, if there is a function $f(x)$ so that $f(n) = a_n$ for every integer n and $\lim_{x\to\infty} f(x) = L$, then
   $$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n = L.$$

   Use this result and tools from Calculus I to compute the limits of the following sequences.

   (a) $\{a_n\}$ where $a_n = \dfrac{n^2 + n + 1}{3n^2 + 2n + 15}$.

   (b) $\{a_n\}$ where $a_n = n^2 \cdot e^{-n}$

   (c) $\{a_n\}$ where $a_n = \dfrac{(\ln(n))^2}{n}$