Worksheet 10 Key

1. a) Yes, all such functions have Taylor series expansions, but the radius of convergence may vary.

b) If $f^{(n+1)}(x)$ exists and is continuous, then the error in $T_n(x)$ is

$$R_n(x) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u)\,du \;;$$

c) No; the Taylor series representation is unique.

2. a) $f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = (-1)^2 2!\, x^{-3}$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$$f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$$

Ratio test: $\displaystyle \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(x-2)} \right| = \frac{|x-2|}{2} < 1$

Converges for $|x-2| < 2$.

Alternatively, $\displaystyle \frac{1}{x} = \frac{1}{2+x-2} = \frac{1}{2} \cdot \frac{1}{1+\left(\frac{x-2}{2}\right)}$

$$= \sum_{n=0}^{\infty} \frac{\left(-\frac{x-2}{2}\right)^n}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n .$$

b) $f(x) = \cos(x)$ $\quad f''(x) = -\cos(x)$ $\quad f^{(4)}(x) = \cos(x)$

$f'(x) = -\sin(x)$ $\quad f'''(x) = \sin(x)$

$$f^{(n)}(\pi) = \begin{cases} -1 & \text{if } n \equiv 0 \bmod 4 \\ 1 & \text{if } n \equiv 2 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (x-\pi)^{2n} = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \dots$$

Ratio test: $\displaystyle \lim_{n\to\infty} \left| \frac{(x-\pi)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(x-\pi)^{2n}} \right| = \lim_{n\to\infty} \frac{|x-\pi|^2}{(2n+1)(2n+2)} = 0 < 1$

Converges everywhere.

3\. As in 2b, ... $f^{(n)}(0) = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 4 \\ -1 & \text{if } n \equiv 2 \bmod 4 \\ 0 & \text{, otherwise} \end{cases}$

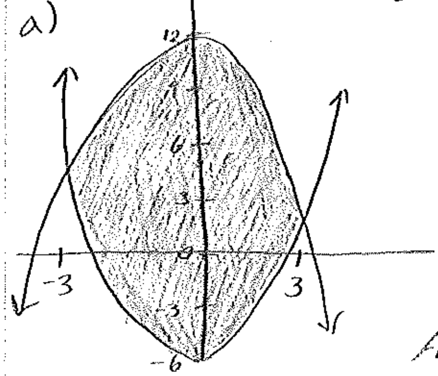
$$\cos(x) = \cos(0) - \sin(0)x - \frac{\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 + \frac{\cos(0)}{4!}x^4 + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Again, this converges on all $x \in \mathbb{R}$.

5\. a)



Intersection: $12 - x^2 = x^2 - 6$

$$18 = 2x^2$$
$$9 = x^2$$
$$x = \pm 3$$

$$\text{Area} = \int_{-3}^{3} (12 - x^2) - (x^2 - 6)\, dx$$

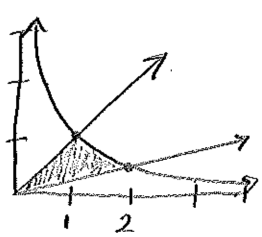
$$= \int_{-3}^{3} 18 - 2x^2\, dx$$

$$= \left[18x - \tfrac{2}{3}x^3 \right]_{-3}^{3}$$

$$= (54 - 18) - (-54 + 18)$$

$$= 72 \text{ units}^2$$

b)



$\frac{1}{x} = x$ for $x = 1$

$\frac{1}{x} = \frac{1}{4}x$ for $x = 2$

$$\text{Area} = \int_0^1 x - \tfrac{1}{4}x\, dx + \int_1^2 \tfrac{1}{x} - \tfrac{1}{4}x\, dx$$

$$= \left[\tfrac{3}{8}x^2 \right]_0^1 + \left[\ln(x) - \tfrac{1}{8}x^2 \right]_1^2$$

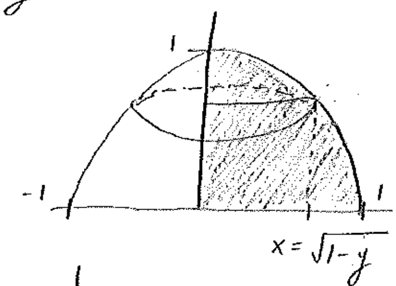
$$= 3/8 + \left[(\ln(2) - 1/2) - (-1/8) \right]$$

$$= 3/8 + \ln(2) - 3/8$$

$$= \ln(2)$$

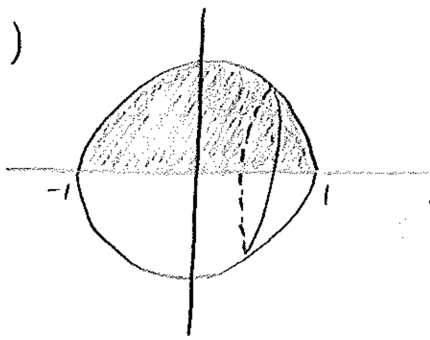
6. a) $V = \int_a^b A(y)\,dy$

b) If the cross-sections are discs with area $\pi(1-y)$, then the radius of each disc is $x = \sqrt{1-y}$. So $y = 1-x^2$ is the object we are looking at.



Cross-sectional area is $\pi x^2 = \pi(1-y)$.

7. a)



$y = 1-x^2 \quad A(x) = \pi(1-x^2)^2$
$= \pi(1-2x^2+x^4)$

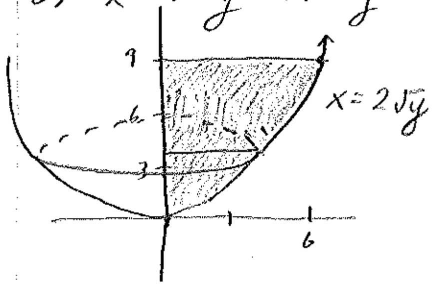
$V = \int_{-1}^{1} \pi(1-2x^2+x^4)\,dx$

$= \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1}$

$= \pi\left[\left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]$

$= \frac{\pi 16}{15} \approx 3.35103$

b) $x = 2\sqrt{y} \iff y = \frac{1}{4}x^2$

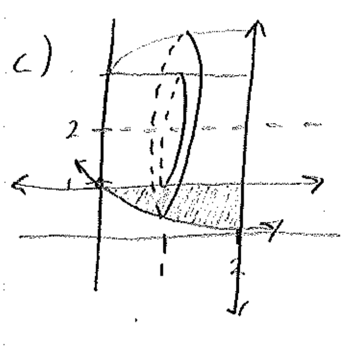


$A(y) = 2x^2 = \pi(2\sqrt{y})^2 = 4\pi y$

$V = \int_0^9 4\pi y\,dy$

$= \left[2\pi y^2\right]_0^9 = 162\pi$

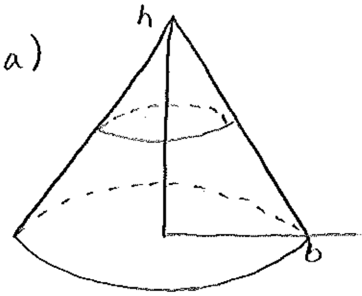
c)



Cross sectional area is
$A(x) = \pi(2-e^{-x})^2 - \pi(1)^2$
$= \pi(3 - 4e^{-x} + e^{-2x})$

$V = \pi\int_0^2 3 - 4e^{-x} + e^{-2x}\,dx$
$= \pi\left[3x + 4e^{-x} - \frac{1}{2}e^{-2x}\right]_0^2$
$= \pi\left[\left(6+4e^{-2}-\frac{1}{2}e^{-4}\right) - \left(4-\frac{1}{2}\right)\right]$
≈ 9.52588

8. a)



Line through $(0,h)$ and $(b,0)$:

$$y = -\frac{h}{b}x + h$$

$$x = -\frac{b}{h}y + b$$

$$A(y) = \pi\left(-\frac{b}{h}y + b\right)^2 = \pi\left(\left(\frac{b}{h}\right)^2 y^2 - 2\frac{b^2}{h}y + b^2\right)$$

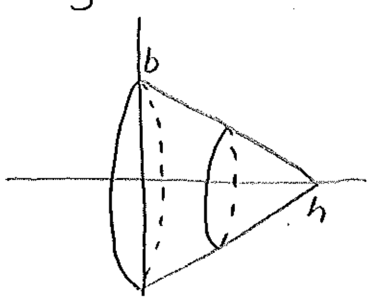
$$V = \pi\int_0^h \left(\frac{b}{h}\right)^2 y^2 - 2\frac{b^2}{h}y + b^2 \, dy$$

$$= \pi\left[\frac{b^2}{3h^2}y^3 - \frac{b^2}{h}y^2 + b^2 y\right]_0^h$$

$$= \pi\left[\left(\frac{b^2h}{3} - b^2h + b^2h\right) - 0\right]$$

$$= \frac{\pi b^2 h}{3}$$

b)



Same argument, but interchange x and y.