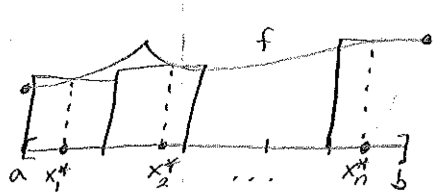
Worksheet 11 Key

1. a)



$Avg_n = \dfrac{f(x_1^*) + \ldots + f(x_n^*)}{n}$

If all subintervals are of size $\Delta x = \frac{b-a}{n}$, then $n = \frac{b-a}{\Delta x}$ and we have:

$$Avg_n = \frac{\sum_{i=1}^{n} f(x_i^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Then $\lim_{n\to\infty} Avg_n = \frac{1}{b-a}\int_a^b f(x)\,dx$.

b) If $f(x)$ is continuous on $[a,b]$, then there exists a value $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

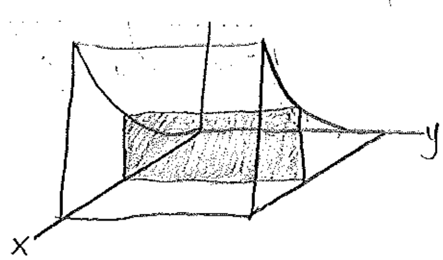
Proof. Let $F(x) = \int_a^x f(t)\,dt$. Then by Fundamental Theorem of Calculus, $F'(x) = f(x)$. Since $F(a)=0$ and $F(b) = \int_a^b f(x)\,dx$, by the Mean Value Theorem there exists $c \in (a,b)$ so that

$$f(c) = F'(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

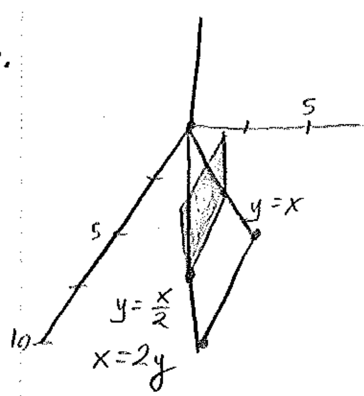
c) Volume $= \int A(x)\,dx$

2. $A(x) = x^2$.

$V = \int_0^1 x^2\,dx = [\frac{1}{3}x^3]_0^1 = 1/3$



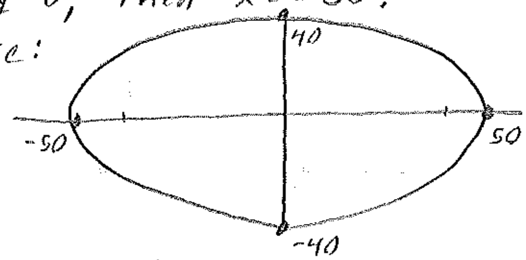
3.

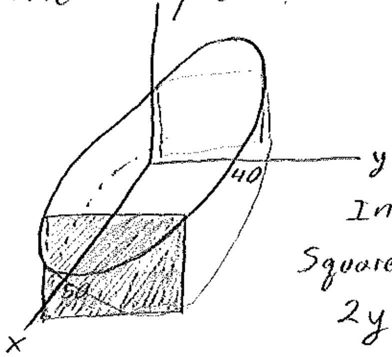


$A(y) = (2y-y)^2 = y^2$

$V = \int_0^5 y^2\,dy = [\frac{1}{3}y^3]_0^5 = \frac{125}{3}$

4. If $x=0$, then $y=\pm 40$. If $y=0$, then $x=\pm 50$. So the ellipse looks like:





Integrate over x.
Squares have side length

$$2y = 2\sqrt{1-\left(\frac{x}{50}\right)^2}$$

$$A(x) = (2y)^2 = 4\left(1-\left(\frac{x}{50}\right)^2\right) = 4 - \frac{x^2}{625}$$

$$V = \int_{-50}^{50} 4 - \frac{x^2}{625}\, dx$$

5. a) $f(x) = x^2+1$ on $[1,3]$

$$Avg = \frac{1}{3-1}\int_1^3 x^2+1\, dx$$

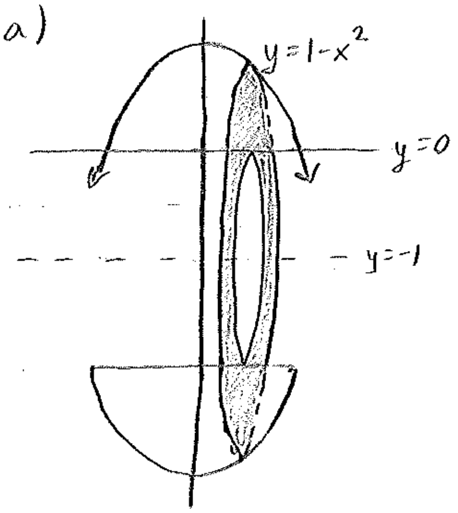
b) $g(x) = e^x$ on $[-1,1]$

$$Avg = \frac{1}{1-(-1)}\int_{-1}^{1} e^x\, dx$$

c) $h(x) = \dfrac{3}{x^2+1}$ on $[0,1]$

$$Avg = \frac{1}{1-0}\int_0^1 \frac{3}{x^2+1}\, dx$$

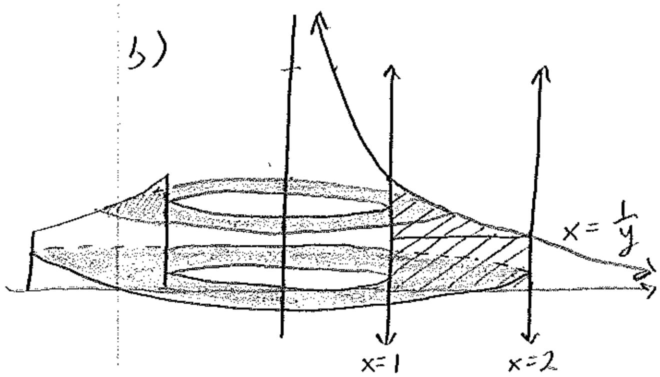
6. a)



$$A(x) = \pi\left(1-x^2-(-1)\right)^2 - \pi\left(0-(-1)\right)^2 = \pi(2-x^2)^2 - \pi$$

$$V = \int_{-1}^{1} \pi(2-x^2)^2 - \pi \, dx$$

b)

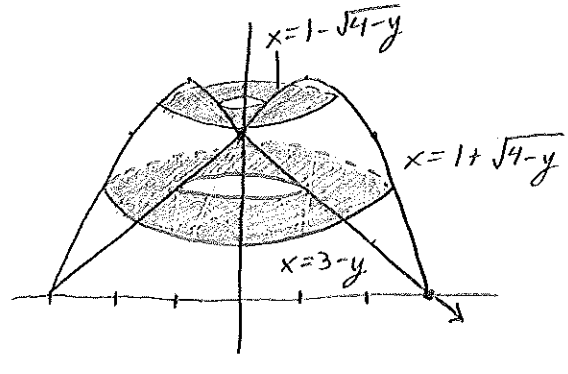


split into two integrals.

$$V = \int_0^{1/2} \pi 2^2 - \pi 1^2 \, dy + \int_{1/2}^{1} \pi\left(\frac{1}{y}\right)^2 - \pi 1^2 \, dy$$

c) $y = 3 + 2x - x^2 = (3-x)(1+x) = -(x-3)(x+1)$

$x + y = 3 \Rightarrow x = 3 - y$



Complete the square to solve for x.

$$-y = x^2 - 2x - 3$$
$$4-y = x^2 - 2x + 1$$
$$4-y = (x-1)^2$$
$$\pm\sqrt{4-y} = x-1$$
$$x = 1 \pm \sqrt{4-y}$$

$$V = \int_0^3 \pi\left(1+\sqrt{4-y}\right)^2 - \pi(3-y)^2 \, dy + \int_3^4 \pi\left(1+\sqrt{4-y}\right)^2 - \pi\left(1-\sqrt{4-y}\right)^2 dy$$