Worksheet 15 Key

1. a) $\sec^2(\theta) = \dfrac{1}{\cos^2(\theta)} = \dfrac{\sin^2(\theta)+\cos^2(\theta)}{\cos^2(\theta)} = \tan^2(\theta)+1.$

b) Given $x = a\sin(\theta)$, we have

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2(\theta)} = a\sqrt{1-\sin^2(\theta)} = a\sqrt{\cos^2(\theta)}$$
$$= a\,|\cos(\theta)| = a\cdot\cos(\theta) \quad \text{since } \cos(\theta)\geq 0 \text{ when } |\theta| \leq \tfrac{\pi}{2}.$$

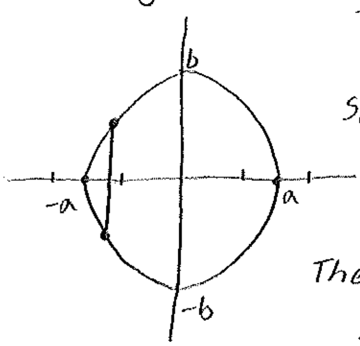
c) Given $x = a\tan\theta$, we have

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2(\theta)} = a\sqrt{1+\tan^2(\theta)} = a\sqrt{\sec^2(\theta)}$$
$$= a\,|\sec(\theta)| = a\sec(\theta) \quad \text{since } \sec(\theta) > 0 \text{ when } |\theta| < \tfrac{\pi}{2}.$$

d) Given $x = a\sec(\theta)$, we have

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2(\theta)-a^2} = a\sqrt{\sec^2(\theta)-1} = a\sqrt{\tan^2(\theta)}$$
$$= a\,|\tan(\theta)| = a\tan(\theta) \quad \text{since } \tan(\theta)>0 \text{ on } [0,\tfrac{\pi}{2}) \text{ and } [\pi,\tfrac{3\pi}{2}).$$

3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. To plot, set $x=0$ and see $y=\pm b$.
Set $y=0$ and see $x=\pm a$.



Solving for y, we get the hemispheres
$$y = \pm\sqrt{\frac{a^2b^2-b^2x^2}{a^2}}.$$

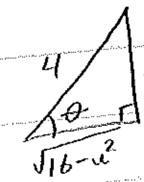
The length of a cross section at x is
$$L(x) = 2\sqrt{\frac{a^2b^2-b^2x^2}{a^2}} = \frac{2b}{a}\sqrt{a^2-x^2}.$$

Then the area is $\displaystyle\int_{-a}^{a} \frac{2b}{a}\sqrt{a^2-x^2}\,dx$. Put $x = a\sin(\theta)$. (So $\theta = \sin^{-1}(\frac{x}{a})$.)
Then $dx = a\cos(\theta)\,d\theta$.

Then the integral becomes $\displaystyle\int_{-a}^{a} \frac{2b}{a}\, a\cos(\theta)\, a\cos(\theta)\,d\theta$

$$= 2ab\int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \cos^2(\theta)\,d\theta = 2ab\left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2ab\left[\left(\frac{\pi}{4}\right)-\left(-\frac{\pi}{4}\right)\right] = \pi ab.$$

2.(a) $\int_0^2 \frac{u^3}{\sqrt{16-u^2}}\,du$



So, $\sqrt{16-u^2} = \sqrt{4^2-u^2}$ So, $a = 4$

we will use the substitution:

$u = 4\sin\theta$ $du = 4\cos\theta$ and $\sqrt{4^2-u^2} = 4\cos\theta$

$\theta = \sin^{-1}\left(\frac{u}{4}\right)$

$$\int_0^{\pi/6} \frac{u^3}{\sqrt{16-u^2}}\,du = \int_0^{\frac{\pi}{6}} \frac{4^3\sin^3\theta}{4\cos\theta}\cdot 4\cos\theta\,d\theta = \int_0^{\frac{\pi}{6}} 4^3\sin^3\theta\,d\theta$$

$$= 64\int_0^{\frac{\pi}{6}} \sin^3\theta\,d\theta$$

$$= 64\int_0^{\frac{\pi}{6}} (1-\cos^2\theta)\sin\theta\,d\theta$$

Using identity, $\sin^2\theta = 1-\cos^2\theta$

Put $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$

$$= -64\int_1^{\frac{\sqrt{3}}{2}} (1-u^2)\,du = 64\left(\frac{u^3}{3} - u\right)\Big|_1^{\frac{\sqrt{3}}{2}}$$

$$= 64\left[\frac{\left(\frac{\sqrt{3}}{2}\right)^3}{3} - \frac{\sqrt{3}}{2} - \left(\frac{1}{3} - 1\right)\right]$$

$$\approx 1.09745$$

(2b) $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

$\sqrt{25-x^2} = \sqrt{5^2-x^2} \Rightarrow x = 5\sin\theta$ $dx = 5\cos\theta\,d\theta$

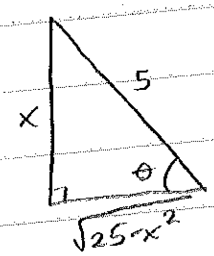
$\sqrt{5^2-x^2} = 5\cos\theta$

$$\int \frac{1}{x^2\sqrt{25-x^2}} = \int \frac{1}{25\sin^2\theta\, 5\cos\theta}\cdot 5\cos\theta\,d\theta$$

$$= \int \frac{1}{25\sin^2\theta}\,d\theta = \frac{1}{25}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{25}\int \csc^2\theta\,d\theta = -\frac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\left(\frac{\sqrt{25-x^2}}{x}\right) + C$$



2c) $\int \sqrt{5+4x-x^2}\,dx$

$5+4x-x^2 \Longrightarrow$ complete the square

$-(x^2-4x-5) = -(x^2-4x+4-9) = -((x-2)^2-9) = 9-(x-2)^2$

$\int \sqrt{9-(x-2)^2}\,dx$ $\quad u = x-2$, $du = dx$

$\int \sqrt{9-u^2}\,du$ $\quad u = 3\sin\theta$, $du = 3\cos\theta\,d\theta$ $\quad \sqrt{9-u^2} = 3\cos\theta$

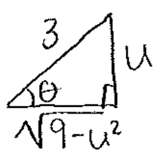
$$\int 3\cos\theta \cdot 3\cos\theta\,d\theta = \int 9\cos^2\theta\,d\theta = 9\int \cos^2\theta\,d\theta$$

$$= 9\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{9}{2}\int [1+\cos 2\theta]\,d\theta = \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

double-angle formula

$$= \frac{9}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C = \frac{9}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

we know
$u = 3\sin\theta$
$\sin\theta = u/3 = \frac{opp}{hyp}$



so $\cos\theta = \frac{\sqrt{9-u^2}}{3}$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{u}{3}\right) + \frac{u}{3} \cdot \frac{\sqrt{9-u^2}}{3}\right] + C$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{3} \cdot \frac{\sqrt{9-(x-2)^2}}{3}\right] + C = \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{(x-2)\sqrt{9-(x-2)^2}}{9}\right] + C$$

2) d) $\displaystyle\int \frac{x^3}{\sqrt{64+x^2}}\,dx$

$x = 8\tan\theta$

$dx = 8\sec^2\theta \, d\theta$

$\sqrt{64+x^2} = \sqrt{8^2+x^2} = 8\sec\theta$

$$\int \frac{512\tan^3(\theta)}{8\sec(\theta)} \cdot 8\sec^2(\theta)\,d\theta$$

$$= \int 512\tan^3(\theta)\sec(\theta)\,d\theta$$

$$= \int 512\tan^2(\theta)\tan(\theta)\sec(\theta)\,d\theta$$

$$= \int 512(\sec^2(\theta)-1)\tan(\theta)\sec(\theta)\,d\theta$$

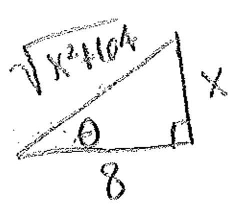
$u = \sec(\theta)$

$du = \sec(\theta)\tan(\theta)\,d\theta$

$$= \int 512(u^2-1)\,du = 512\int (u^2-1)\,du = 512\left[\frac{u^3}{3} - u\right] + C$$

$$= 512\left[\frac{\sec^3(\theta)}{3} - \sec^3(\theta)\right] + C$$

Convert back to x



$x = 8\tan\theta$

$\tan\theta = x/8 = \text{opp}/\text{adj}$

$\sec\theta = \text{hyp}/\text{adj} = \sqrt{x^2+64}/8$

$$512\left[\frac{\sec^3(\theta)}{3}-\sec^3(\theta)\right]+C$$

$$=512\left[\left(\frac{\sqrt{x^2+64}}{8}\right)^3\cdot\frac{1}{3}-\frac{\sqrt{x^2+64}}{8}\right]+C$$

$$=512\left[\frac{(x^2+64)(\sqrt{x^2+64})}{512}\cdot\frac{1}{3}-\frac{\sqrt{x^2+64}}{8}\right]+C$$

$$=\frac{(x^2+64)(\sqrt{x^2+64})}{3}-64\sqrt{x^2+64}\quad+C$$

e) $\int_0^1\sqrt{x^2+1}\,dx$ $\quad x=\tan\theta \quad \sqrt{x^2+1}=\sec\theta$

$dx=\sec^2\theta\,d\theta$

Limits of integration: $0=\tan(\theta) \Rightarrow \theta=0$

$1=\tan(\theta)\Rightarrow \theta=\pi/4$

$$=\int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta$$

$u=\sec\theta \quad dv=\sec^2\theta\,d\theta$

$du=\sec\theta\tan\theta\,d\theta \quad v=\tan\theta$

$$=\sec\theta\tan\theta\Big|_0^{\pi/4}-\int_0^{\pi/4}\tan^2\theta\sec\theta\,d\theta=\sec\theta\tan\theta-\int_0^{\pi/4}(\sec^2\theta-1)\sec\theta\,d\theta$$

$$=\sec\theta\tan\theta\Big|_0^{\pi/4}-\int_0^{\pi/4}\sec^3\theta-\sec\theta\,d\theta=\sec\theta\tan\theta\Big|_0^{\pi/4}-\int_0^{\pi/4}\sec^3\theta\,d\theta+\int_0^{\pi/4}\sec\theta\,d\theta$$

So we have

$$\int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta=\int_0^{\pi/4}\sec^3\theta\,d\theta=\sec\theta\tan\theta\Big|_0^{\pi/4}-\int_0^{\pi/4}\sec^3\theta\,d\theta+\int_0^{\pi/4}\sec\theta\,d\theta$$

Add $\int_0^{\pi/4} \sec^3\theta \, d\theta$ to both sides

$$2\int_0^{\pi/4} \sec^3\theta \, d\theta = \sec\theta\tan\theta \Big|_0^{\pi/4} + \int_0^{\pi/4} \sec\theta \, d\theta$$

$$\int_0^{\pi/4} \sec^3\theta \, d\theta = \frac{\sec\theta\tan\theta}{2}\Big|_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \frac{\sec\theta\tan\theta}{2}\Big|_0^{\pi/4} + \frac{1}{2}\left[\ln|\sec\theta + \tan\theta|\right]\Big|_0^{\pi/4}$$

$$= \frac{\sec(\pi/4)\tan(\pi/4)}{2} - \frac{\sec(0)\tan(0)}{2} + \frac{1}{2}\left[\ln|\sec(\pi/4) + \tan(\pi/4)| + \ln|\sec(0) + \tan(0)|\right]$$

$$= \frac{\sqrt{2}\cdot 1}{2} - \frac{1\cdot 0}{2} + \frac{1}{2}\left[\ln|\sqrt{2}+1| + \ln|1\cdot 0|\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\left[\ln|\sqrt{2}+1| + \cancelto{0}{\ln|1|}\right] = \frac{\sqrt{2}}{2} + \frac{\ln|\sqrt{2}+1|}{2}$$

f) Put $u = x^2+1$. Then $du = 2x\,dx$. We then have

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du$$

$$= u^{1/2} = \sqrt{x^2+1}.$$