Worksheet 16 Key

1. a) The Taylor series expansion for any polynomial $f(x)$ is just $f(x)$. In this case, $T(x) = f(x) = x - x^3$. The radius of convergence is ∞. If you use the Taylor series formula to prove this, you will get $T(x) = 6 - 11(x+2) + 6(x+2)^2 - (x+2)^3 = x - x^3$.

b) $f(x) = x^{-1}$, centered at 1.

$f'(x) = -1x^{-2}$

$f''(x) = (-1)^2 \cdot 2x^{-3}$

$f'''(x) = (-1)^3 \cdot 3!\, x^{-4}$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$f^{(n)}(1) = (-1)^n n!$

Then $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$

Alternatively, use geometric series:

$$\frac{1}{x} = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} (-(x-1))^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

as long as $|x-1| < 1 \Rightarrow x$ in $(0,2)$.

c) $f(x) = e^{-x^2} + e^{x^2}$ about 0. Using the Maclaurin expansion $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all real x, we have

$$T(x) = \sum_{n=0}^{\infty} \left(\frac{(-x^2)^n}{n!} + \frac{(x^2)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{n!} x^{2n}$$

with radius ∞.

d) $f(x) = \ln(1+x^2)$ about 0. Using the Maclaurin expansion $\ln(1-x) = -\sum_{n=1}^{\infty} x^n/n$ for $|x| < 1$, we have

$$T(x) = f(x) = \ln(1-(-x^2)) = -\sum_{n=1}^{\infty} (-x^2)^n/n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n} \quad \text{for } x \text{ in } (-1,1).$$

2. Natural length of .2 m. Common sense dictates that to stretch the spring will require more force at higher lengths, so $W_2 > W_1$.

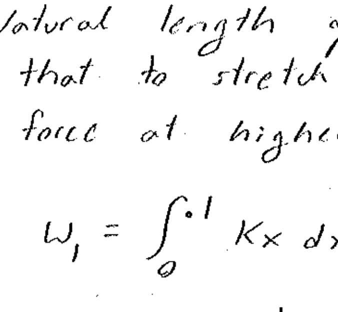
$$W_1 = \int_0^{.1} Kx\,dx \quad \text{and} \quad W_2 = \int_{.1}^{.2} Kx\,dx$$

$$= \frac{Kx^2}{2}\Big|_0^{.1} \qquad\qquad = \frac{Kx^2}{2}\Big|_{.1}^{.2}$$

$$= .005\,K \qquad\qquad = .015\,K.$$

So regardless of spring constant K, our hypothesis that $W_2 > W_1$ was correct.

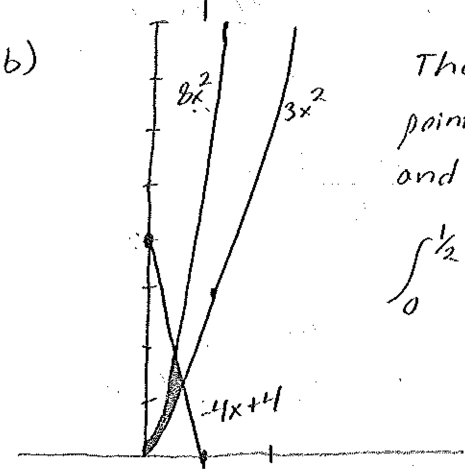
3. a) $y = \sqrt{25 - x^2}$ is the upper half of a circle, radius 5.



Using the disc method, cross-sections are circles of radius $\sqrt{25-x^2}$. So the volume is given by

$$\pi \int_{-5}^{5} \left(\sqrt{25-x^2}\right)^2 dx = \frac{500\pi}{3}.$$

b)

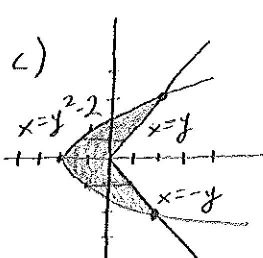


The hint tells us that the intersection points of the curves are at $\frac{1}{2}$ and $\frac{2}{3}$. Then the area is

$$\int_0^{1/2} 8x^2 - 3x^2\,dx + \int_{1/2}^{2/3} -4x+4-3x^2\,dx$$

$$= 17/54$$

c)

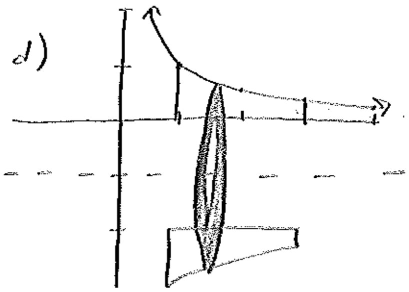


Integrating over y will be easier.

$$\int_{-2}^{0} (-y) - (y^2-2)\,dy + \int_0^2 y - (y^2-2)\,dy$$

$$= 20/3$$

3. d)

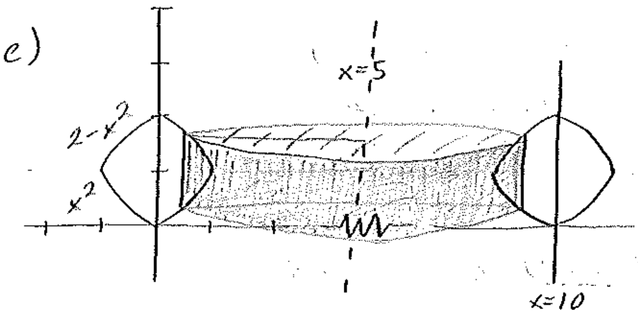


Cross-sections are washers with outer radius $\frac{1}{x}-(-1)$ and inner radius $0-(-1)$. Area of a cross section at x is

$$A(x) = \pi\left(\tfrac{1}{x}+1\right)^2 - \pi(1)^2.$$

Volume is $\pi \int_1^3 \left(\tfrac{1}{x}+1\right)^2 - 1 \, dx$

e)

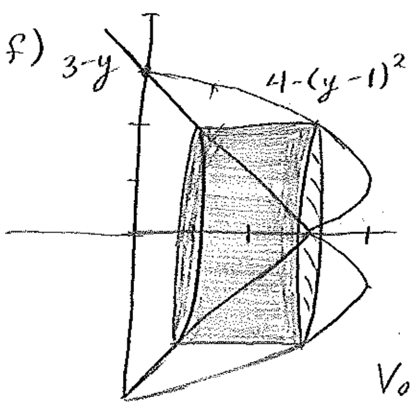


Shell with left edge at x has radius $5-x$ and height $(2-x^2)-x^2$. Surface area:

$$A(x) = 2\pi(5-x)(2-2x^2)$$

Volume is $\int_{-1}^{1} 2\pi(5-x)(2-2x^2)$

f)



Shells are much easier here. The shell with top edge at y has radius y and height $(4-(y-1)^2-(3-y))$, thus surface area $A(y) = 2\pi y\,(3y-y^2)$.

Volume is $\int_0^3 2\pi y\,(3y-y^2)\,dy$.

4. a) Let $u=t$, $dv=\sin(t)\,dt$. Then $du=dt$, $v=-\cos(t)$,

$$\int t\sin(t)\,dt = -t\cos(t) + \int \cos(t)\,dt$$
$$= -t\cos(t) + \sin(t) + C$$

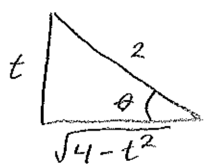
b) See 2b on Worksheet 14.

c) $$\int \sin(t)\cot(t)\,dt = \int \sin(t)\cdot\frac{\cos(t)}{\sin(t)}\,dt$$
$$= \int \cos(t)\,dt$$
$$= \sin(t) + C$$

4. d) Let $u = \sin(t)$. Then $du = \cos(t)\,dt$.

$$\int \sin^2(t)\cos^3(t)\,dt = \int \sin^2(t)\left(1-\sin^2(t)\right)\cos(t)\,dt$$
$$= \int u^2(1-u^2)\,du$$
$$= \int u^2 - u^4\,du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + c$$
$$= \frac{\sin^3(t)}{3} - \frac{\sin^5(t)}{5} + c.$$

e)



$\sin(\theta) = \frac{t}{2} \Rightarrow t = 2\sin(\theta)$

$dt = 2\cos(\theta)\,d\theta$

$\cos(\theta) = \frac{\sqrt{4-t^2}}{2} \Rightarrow \sqrt{4-t^2} = 2\cos(\theta)$

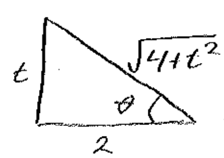
Then $\displaystyle\int \frac{t}{\sqrt{4-t^2}}\,dt = \int \frac{2\sin(\theta)}{2\cos(\theta)} \cdot 2\cos(\theta)\,d\theta$

$$= \int 2\sin(\theta)\,d\theta$$
$$= -2\cos(\theta) + c$$
$$= -\sqrt{4-t^2} + c$$

f) $\displaystyle\int \tan^2(t)\,dt = \int \sec^2(t)\,dt - \int 1\,dt$

$$= \tan(t) - t + c$$

g)



$\tan(\theta) = \frac{t}{2} \Rightarrow t = 2\tan(\theta)$

$dt = 2\sec^2(\theta)\,d\theta$

$\sec(\theta) = \frac{\sqrt{4+t^2}}{2} \Rightarrow \sqrt{4+t^2} = 2\sec(\theta)$

Then $\displaystyle\int \frac{1}{\sqrt{4+t^2}}\,dt = \int \frac{1}{2\sec(\theta)} \cdot 2\sec^2(\theta)\,d\theta$

$$= \int \sec(\theta)\,d\theta$$
$$= \ln\left|\sec(\theta) + \tan(\theta)\right| + c$$
$$= \ln\left|\frac{\sqrt{4+t^2}}{2} + \frac{t}{2}\right| + c$$

4. h) Let $u = \sec(\theta)$. Then $du = \sec(\theta)\tan(\theta)\,d\theta$.

$$\int \tan^3(\theta)\sec^3(\theta)\,d\theta = \int (\sec^2(\theta)-1)\sec^2(\theta)\sec(\theta)\tan(\theta)\,d\theta$$
$$= \int (u^2-1)u^2\,du$$
$$= \int u^4 - u^2\,du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$
$$= \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + C$$

5. Average value of the function $T(t)$ from $30 \le t \le 60$:

$$\frac{1}{60-30}\int_{30}^{60} 75 + 125e^{-.02t}\,dt \approx 126.587 \ {}^{o}F$$