MA 114 Worksheet #17: Integration by Partial Fractions

1) a) $\dfrac{3}{\underbrace{x^2+2x+4}_{\text{this is an irreducible polynomial}}}$

Because x^2+2x+4 is an irreducible polynomial, we cannot break this into partial fractions

b) $\dfrac{x}{\underbrace{(x^2+1)}_{\text{this is an irreducible polynomial}}(x+2)(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+1}$

c) $\dfrac{1}{x^2+3x+2} = \dfrac{1}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

d) $\dfrac{3x+1}{(x^2+1)^3(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C_1+D_1x}{x^2+1} + \dfrac{C_2+D_2x}{(x^2+1)^2} + \dfrac{C_3+D_3x}{(x^2+1)^3}$

3) $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx = \int \frac{1}{u^3-u^2}\cdot 6u^5\,du = \int \frac{6u^5}{u^3-u^2}\,du$

$u = \sqrt[6]{x}$

$du = \dfrac{1}{6\sqrt[6]{x^5}}\,dx$

$dx = 6\sqrt[6]{x^5}\,du$

$= 6u^5\,du$

$\displaystyle = \int \frac{6u^5}{u^2(u-1)}\,du = \int \frac{6u^3}{u-1}\,du$

Long division:

$$\begin{array}{r l}
 & 6u^2+6u+6 \\
u-1 \,\big) & 6u^3+0u^2+0u+0 \\
 & 6u^3-6u^2 \\
\hline
 & +6u^2+0u+0 \\
 & 6u^2-6u \\
\hline
 & +6u+0 \\
 & 6u-6 \\
\hline
 & 6
\end{array}$$

So $\int \frac{6u^3}{u-1}\,du = \int \left[6u^2 + 6u + 6 - \frac{6}{u-1}\right] du$

$$= 6\int [u^2 + u + 1]\,du - 6\int \frac{du}{u-1}$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u\right] - 6\ln|u-1| + C$$

$$= 6\left[\frac{(\sqrt[6]{x})^3}{3} + \frac{(\sqrt[6]{x})^2}{2} + \sqrt[6]{x} - \ln\left|\sqrt[6]{x} - 1\right|\right] + C$$

$$= 6\left[\frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} - \ln\left|\sqrt[6]{x} - 1\right|\right] + C$$

use long division.

2(a) $\int \frac{x^3-2x^2+1}{x^3-2x^2}dx$

$$x^3-2x^2 \overline{)\,x^3-2x^2+1\,}\quad 1$$
$$\underline{x^3-2x^2\downarrow}$$
$$0+1$$

$= \int \left[1+\frac{1}{x^3-2x^2}\right]dx$

$= \int dx + \int \frac{dx}{x^2(x-2)} = x + \int \left[\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2}\right]dx$

$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-2} = \frac{A(x)(x-2)}{x^2(x-2)}+\frac{B(x-2)}{x^2(x-2)}+\frac{C(x^2)}{x^2(x-2)} = \frac{1}{x^2(x-2)}$

$Ax^2-2Ax+Bx-2B+Cx^2=1$

$(A+C)x^2+(B-2A)x-2B=0x^2+0x+1$

So we say $-2B=1$, $B=-1/2$; $B-2A=0$, $-1/2-2A=0$, $-2A=1/2$, $A=-1/4$; $A+C=0$, $-1/4+C=0$, $C=1/4$

So we have

$x+\int\left[\frac{-1/4}{x}-\frac{1/2}{x^2}+\frac{1/4}{x-2}\right]dx$

$x+\int\left[\frac{-1/4}{x}+\frac{1/4}{x-2}-\frac{1}{2}x^{-2}\right]dx$

$x-1/4\ln|x|+1/4\ln|x-2|+1/2x+C$

(b) $\int \frac{x-9}{(x+5)(x-2)}dx$

$\frac{A}{(x+5)}+\frac{B}{(x-2)}=\frac{x-9}{(x+5)(x-2)}$

$A(x-2)+B(x+5)=x-9$

$Ax-2A+Bx+5B=x-9$

$(A+B)x-2A+5B=x-9$

$A+B=1$ and $-2A+5B=-9$

$A=1-B$ $\qquad -2(1-B)+5B=-9$

$-2+2B+5B=-2+7B=-9$

$B=-1$ and $A=2$

$\int\frac{x-9}{(x+5)(x-2)}dx = 2\int\frac{1}{x+5}dx + -1\int\frac{1}{x-2}dx$

$= 2\ln(x+5)+-1\ln(x-2)+C$

2(c) $\int_0^1 \frac{x^3+4}{x^2+4}$

$$x^2+4 \overset{\displaystyle x}{\overline{\big)\, x^3+4}}$$
$$\underline{x^3+4x}$$
$$-4x+4$$

So, $\dfrac{x^3+4}{x^2+4} = x + \dfrac{(-4x+4)}{\underbrace{(x^2+4)}_{\text{irreducible}}}$

$$\int_0^1\left[x+\frac{4-4x}{x^2+4}\right]dx = \int_0^1 x\,dx + \int_0^1 \frac{4-4x}{x^2+4}\,dx$$

$$= \int_0^1 x\,dx + \int_0^1 \frac{4\,dx}{x^2+4} - \int_0^1 \frac{4x\,dx}{x^2+4}$$

$u = x^2+4$
$du = 2x\,dx$
$2du = 4x\,dx$

$$= \frac{x^2}{2}\Big|_0^1 + \left[4\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]\Big|_0^1 - \int_4^5 \frac{2du}{u}$$

$$= \frac{x^2}{2}\Big|_0^1 + \left[2\tan^{-1}\left(\frac{x}{2}\right)\right]\Big|_0^1 - \left[2\ln|u|\right]\Big|_4^5$$

$$= \frac{1}{2} + 2\tan^{-1}\left(\frac{1}{2}\right) - \cancelto{0}{2\tan^{-1}(0)} - 2\ln 5 + 2\ln 4$$

$$= \frac{1}{2} + 2\tan^{-1}\left(\frac{1}{2}\right) - 2\ln 5 + 4\ln 2$$

2 ⓓ $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

We start by finding the partial fraction decomposition:

$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Now, solve for A and B:

$A(x+2) + B(x+1) = 1$

$Ax + 2A + Bx + B = 1$

So $A+B=0$ and $2A+B=1$, which gives us

$A=-B$, so $2(-B)+B=1$

$-B=1$

So we have $B=-1$ and $A=1$.

Now going back to our original integral, we have

$$\int \frac{1}{x^2+3x+2}\,dx = \int \frac{1}{x+1} - \frac{1}{x+2}\,dx$$

$$= \ln|x+1| - \ln|x+2| + C$$

2 ⓔ $\displaystyle\int \frac{x^3+1}{(x^2-1)^2}\,dx = \int \frac{x^3+1}{(x+1)^2(x-1)^2}\,dx = \int \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}\,dx$

Solving for A, B, C, and D, we get:

$A((x+1)(x-1)^2) + B(x-1)^2 + C((x+1)^2(x-1)) + D(x+1)^2 = x^3+1$

$A((x+1)(x^2-2x+1)) + B(x^2-2x+1) + C((x^2+2x+1)(x-1)) + D(x^2+2x+1) = x^3+1$

$Ax^3 - 2Ax^2 + Ax + Ax^2 - 2Ax + A + Bx^2 - 2Bx + B + Cx^3 + 2Cx^2 + Cx - Cx^2 - 2Cx - C + Dx^2 + 2Dx + D$
$= x^3+1$

$(A+C)x^3 + (-A+B+C+D)x^2 + (-A-2B-C+2D)x + (A+B-C+D) = x^3+1$

This gives us 4 equations,

(1) $A+C=1$ (2) $-A+B+C+D=0$ (3) $-A-2B-C+2D=0$ (4) $A+B-C+D=1$

$A=-C$ $\quad -C=-A+B+D$

So $A=-A+B+D$, or

$2A = B+D$, so we have (3) $-\frac{1}{2}(B+D)-2B-\frac{1}{2}(B+D)+2D=0$

$$-B-D-2B+2D=0$$

$$-3B+D=0$$

$$D=-3B.$$

Substituting all these values into (4), we get

(4) $A+B-C+D=1$

$$\frac{1}{2}(B+D)+B-\frac{1}{2}(B+D)+D=1$$

$$B-3B=1$$

$$-2B=1$$

$$B=-\frac{1}{2}$$

So $D=\frac{3}{2}$, $\quad 2A=-\frac{1}{2}+\frac{3}{2}=1$

$A=\frac{1}{2}$, and $C=-\frac{1}{2}$

So the original integral is

$$\int \frac{x^3+1}{(x^2-1)^2}\,dx = \int \frac{1}{2(x+1)} - \frac{1}{2(x+1)^2} - \frac{1}{2(x-1)} + \frac{3}{2(x-1)^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1)} - \frac{1}{(x+1)^2} - \frac{1}{2(x-1)} + \frac{3}{(x-1)^2}\,dx$$

$$= \frac{1}{2}\left[\ln|x+1| + \frac{1}{(x+1)} - \ln|x-1| + \frac{3}{(x-1)}\right] + C$$