Remember to always show positive, continuous, decreasing!

Worksheet 18

1. (a) $\int_0^2 \frac{x}{x^2-5x+6}\,dx$

is improper since it is undefined at $x=2$.

$\frac{2}{4-10+6} = \frac{2}{0}$

(b) $\int_1^2 \frac{1}{2x-1}\,dx$ proper

(c) $\int_1^2 \ln(x-1)\,dx$. $\ln(x-1)$ is undefined at $x=1 \Rightarrow \ln(1-1)=\ln(0)$. So, $\int_1^2 \ln(x-1)\,dx$ is improper.

(d) $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ is improper since both endpoints are infinite.

(e) $\int_0^{\pi/2} \sec x\,dx = \int_0^{\pi/2} \frac{1}{\cos(x)}\,dx$ is improper since $\frac{1}{\cos(\pi/2)} = \frac{1}{0}$ is undefined.

2. e) $\int_0^1 \ln(x)\,dx = \lim_{R\to 0} \int_R^1 \ln(x)\,dx$

$u=\ln(x)$ $du=\frac{1}{x}dx$ $dv=dx$ $v=x$

$uv-\int v\,du$

$\lim_{R\to 0}\left[x\ln x - \int_R^1 dx\right] = \lim_{R\to 0} x\ln x - x\Big|_R^1$

$= \lim_{R\to 0}\left[1\ln(1)-1-R\ln(R)-R\right]$

$= \ln(1)-1$

(f) $\int_0^\infty \sin\theta \, d\theta = \lim_{R\to\infty} \int_0^R \sin\theta \, d\theta$

$= \lim_{R\to\infty} \left[-\cos\theta\right]_0^R$

$= \lim_{R\to\infty} \left[-\cos(R) + \cos(0)\right]$

$= \lim_{R\to\infty} \left[-\cos(R) + 1\right]$

Does not converge.

5.(d) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ $\quad \int_2^\infty \frac{1}{x\ln(x)} dx$

show positive, continuous, decreasing!

$= \lim_{R\to\infty} \int_2^R \frac{1}{x\ln(x)} dx$

$u = \ln(x) \quad du = \frac{1}{x} dx$

$= \lim_{R\to\infty} \int_{\ln(2)}^{\ln(R)} \frac{1}{u} du = \lim_{R\to\infty} \left[\ln|u|\right]_{\ln(2)}^{\ln(R)} = \lim_{R\to\infty} \left[\ln|u|\right]_{\ln(2)}^{\ln(R)}$

$= \lim_{R\to\infty} \ln|\ln(R)| - \ln|\ln(2)| = \infty$

So, $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges

(e) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$ $\quad u = x^3+5 \quad du = 3x^2 dx$

$\int_1^\infty \frac{x^2}{x^3+5} dx = \lim_{R\to\infty} \int_1^R \frac{x^2}{x^3+5} dx = \lim_{R\to\infty} \int_6^{R^3+5} \frac{1}{3} \cdot \frac{1}{u} du$

$= \lim_{R\to\infty} \frac{1}{3} \left[\ln|u|\right]_6^{R^3+5} = \lim_{R\to\infty} \frac{1}{3} \left[\ln|R^3+5| - \ln(6)\right] = \infty$

So $\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$ diverges.

MA 114 Worksheet 18 - Improper Integrals & Integral Test

2) a) $\int_{-\infty}^{0} \frac{1}{2x-1}\,dx$

Evaluate $\int_{R}^{0} \frac{1}{2x-1}\,dx = \int_{2R-1}^{-1} \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\ln|u|\Big|_{2R-1}^{-1}$

$u = 2x-1$

$du = 2dx$

$\frac{du}{2} = dx$

$= \frac{1}{2}\cancel{\ln|-1|}^{\,0} - \frac{1}{2}\ln|2R-1|$

$= -\frac{1}{2}\ln|2R-1|$

$\lim_{R\to-\infty}\int_{R}^{0}\frac{1}{2x-1}\,dx = \lim_{R\to-\infty} -\frac{1}{2}\ln|2R-1| = -\infty$

$\boxed{\text{DIVERGES}}$

b) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx$

Evaluate $\int_{-R}^{0} xe^{-x^2}\,dx + \int_{0}^{R} xe^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$-\frac{du}{2} = x\,dx$

$\int_{-R^2}^{0} -\frac{e^u}{2}\,du + \int_{0}^{-R^2} -\frac{e^u}{2}\,du = -\frac{e^u}{2}\Big|_{-R^2}^{0} + -\frac{e^u}{2}\Big|_{0}^{-R^2}$

$= \frac{-e^0}{2} + \frac{e^{-R^2}}{2} - \frac{e^{-R^2}}{2} + \frac{e^0}{2} = 0$

$\lim_{R\to\infty}\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty} 0 = 0$

$\boxed{\text{CONVERGES}}$

c) $\int_0^\infty \frac{1}{x}\,dx$

Evaluate $\int_0^R \frac{1}{x}\,dx = \int_0^1 \frac{1}{x}\,dx + \int_1^R \frac{1}{x}\,dx$

Evaluate $\int_1^R \frac{1}{x}\,dx = \ln|x|\Big|_1^R = \ln|R| - \ln|1| = \ln|R|$, where $\ln|1| \to 0$

$\lim_{R\to\infty} \ln|R| = \infty$ (this means $\int_1^R \frac{1}{x}\,dx$ diverges)

So $\int_0^R \frac{1}{x}\,dx = \int_0^1 \frac{1}{x}\,dx + \int_1^R \frac{1}{x}\,dx$ $\boxed{\text{DIVERGES}}$

d) $\int_{10^{-10}}^\infty \frac{1}{x^2}\,dx$

Evaluate $\int_{10^{-10}}^R \frac{1}{x^2}\,dx = \int_{10^{-10}}^R x^{-2}\,dx = \frac{x^{-1}}{-1}\Big|_{10^{-10}}^R = -\frac{1}{x}\Big|_{10^{-10}}^R$

$= -\frac{1}{R} + \frac{1}{10^{-10}} = -\frac{1}{R} + 10^{10} = 10^{10} - 1/R$

$\lim_{R\to\infty} \int_{10^{-10}}^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty}\left[10^{10} - 1/R\right] = 10^{10}$

$\boxed{\text{CONVERGES}}$

Worksheet 18 Key

3. First, rewrite as a limit: $\lim_{R\to\infty} \int_1^R x^{-p}\,dx$.

If $p=1$, $\lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_1^R$

$= \lim_{R\to\infty} \ln|R| = \infty$

If $p>1$, $\lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^R$

$= \lim_{R\to\infty} \frac{1}{1-p}\left[\frac{1}{R^{p-1}} - 1\right]$

$= \frac{1}{p-1}$

If $p<1$, $\lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{1}{1-p}\left[x^{1-p}\right]_1^R$

$= \lim_{R\to\infty} \frac{1}{1-p}\left[R^{1-p} - 1\right] = \infty$.

4. So by the integral test, the series converges for $p>1$ and diverges for $p\le 1$.

5. a) $\lim_{R\to\infty} \int_1^R \frac{5}{x^{3/2}}\,dx = \lim_{R\to\infty} 5\int_1^R x^{-3/2}\,dx$

$= \lim_{R\to\infty} 5\left[-\frac{2}{x^{1/2}}\right]_1^R$

$= \lim_{R\to\infty} 5\left[\frac{-2}{\sqrt{R}} + 2\right] = 10$

Then the series $\sum_{n=1}^{\infty} 5/n^{3/2}$ converges.

b) $\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ Solve for A, B, C.

$1 = A(x^2+1) + (Bx+C)x$

$1 = (A+B)x^2 + Cx + A$

Then $A=1$, $B=-1$, $C=0$.

$\lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx + \lim_{R\to\infty} \int_1^R \frac{-x}{x^2+1}\,dx = \lim_{R\to\infty}\left(\left[\ln|x|\right]_1^R + \left[-\frac{1}{2}\ln|x^2+1|\right]_1^R\right)$

$= \lim_{R\to\infty}\left(\ln|R| - \frac{1}{2}\left[\ln|R^2+1| - \ln|2|\right]\right)$

$$= \lim_{R \to \infty} \left(\ln\left(\frac{|R|}{\sqrt{R^2+1}}\right) + \ln\sqrt{2} \right)$$

$$= \ln\sqrt{2}$$

So the series converges by integral test.

c) Put $u = -x^2$. Then $du = -2x\,dx$.

$$\lim_{R \to \infty} \int_1^R x e^{-x^2}\,dx = \lim_{R \to \infty} \int_{-1}^{-R^2} -\frac{1}{2} e^u\,du$$

$$= \lim_{R \to \infty} -\frac{1}{2}\left[e^{-R^2} - e^{-1}\right]$$

$$= \frac{1}{2e}$$

So the series converges by integral test.