Worksheet 19 Key

1. a) Let $f(x)$ be a continuous function on $[a,b]$, and $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ a partition of $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$. Then the distance along f through points $(x_i, f(x_i))$ is approximated by

$$\sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$\approx \sum_{i=1}^{n} \sqrt{1 + f'(x_i)^2}\, \Delta x$$

b) The limit of the above Riemann sum is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

c) The curve for the top half of a circle of radius r is given by $f(x) = \sqrt{r^2 - x^2}$.

The arc length is then

$$\int_{-r}^{r} \sqrt{1 + f'(x)^2}\, dx = \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$=$ trig stuff happens ... $= \pi r$

Then the total circumference is $2\pi r$.

d) $f(x) = \sin(x)$ on $[0, 2\pi]$.

$$\int_0^{2\pi} \sqrt{1 + \cos^2(x)}\, dx \approx 7.6404$$

3. $f(x) = \frac{2}{3}(x^2 - 1)^{3/2}$ on $[1,3]$, $f'(x) = 2x\,(x^2-1)^{1/2}$,

$f'(x)^2 = 4x^2(x^2 - 1) = 4x^4 - 4x^2$.

$$\int_1^3 \sqrt{1 + 4x^4 - 4x^2}\, dx = 15\tfrac{1}{3}$$

M1 114 Worksheet 19 - Arc Length

2) a) $y = \ln(\sec x) \qquad 0 \leq x \leq \pi/4$

$$y' = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)|$$

$$= \ln|\sqrt{2} + 1| - \cancel{\ln|1+0|}^{\,0} = \ln|\sqrt{2}+1|$$

b) $y = \dfrac{x^2}{2} \qquad [0,1]$

$y' = x$

$$S = \int_0^1 \sqrt{1+x^2}\, dx = \int_0^{\pi/4} \sec\theta \sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^3\theta\, d\theta$$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$

see previous worksheets for this integral

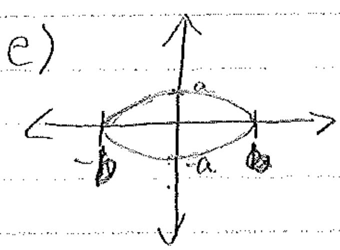
5. a) $\int_0^1 \sqrt{1+(2xe^{x^2})^2}\,dx$ $f(x)=e^{x^2}$
$f'(x)=2xe^{x^2}$

b) $\int_0^{\pi/4} \sqrt{1+(\sec^2 x)^2}\,dx$ $f(x)=\tan x$
$f'(x)=\sec^2 x$

c) $\int_1^4 \sqrt{1+(1+3y^2)^2}\,dy$ $f(y)=y+y^3$
$f'(y)=1+3y^2$

d) $\int_0^1 \sqrt{1+\frac{1}{1+x^2}}\,dx$ $f(x)=\arctan x$
$f'(x)=\frac{1}{1+x^2}$

e)



$\frac{x^2}{b^2}=1-\frac{y^2}{a^2}$

$x^2=b^2-\frac{b^2y^2}{a^2}$

$g(y)=\sqrt{b^2-\frac{b^2y^2}{a^2}}$

$2\int_{-a}^{a}\sqrt{1+\left(\frac{1}{2}\left(b^2-\frac{b^2y^2}{a^2}\right)^{-1/2}\left(\frac{-2b^2}{a^2}\right)y\right)^2}\,dy$

$g'(y)=\frac{1}{2}\left(b^2-\frac{b^2y^2}{a^2}\right)^{-1/2}\cdot\frac{-2b^2}{a^2}y$