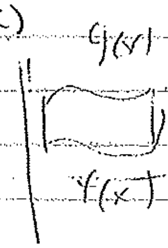
Math 114: Worksheet 21

Center of Mass

1. Conceptual Understanding:

(a) Write down the formula for the coordinates of the centroid of a plate with constant density bounded between $x=a$ and $x=b$, $f(x)$, and $g(x)$ as in the figure to the right.



$$M_y = \rho \int_a^b x(g(x)-f(x))\,dx$$

$$M_x = \frac{1}{2}\rho \int_a^b g(x)^2 - f(x)^2\,dx$$

$$Y_{CM} = \frac{M_x}{M}$$

$$X_{CM} = M_y/M \qquad Y_{CM} = M_x/M$$

$$M = \rho A = \rho \int_a^b g(x) - f(x)\,dx$$

(b) Write down the formulas for the coordinates of the centroid of a plate with constant density bounded between $y=c$, $y=d$, $f(y)$, and $g(y)$ as in the figure to the right.

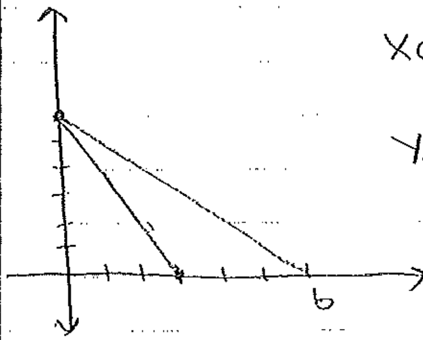
$$M_x = \rho \int_c^d y(f(y)-g(y))\,dy$$

$$M_y = \frac{1}{2}\rho \int_c^d f(y)^2 - g(y)^2\,dy$$

$$X_{CM} = M_y/M \qquad Y_{CM} = M_x/M$$

$$M = \rho A = \rho \int_c^d f(y) - g(y)\,dy$$

3. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3,0)$, $(b,0)$, and $(0,6)$, where $b>3$. Find the center of mass.

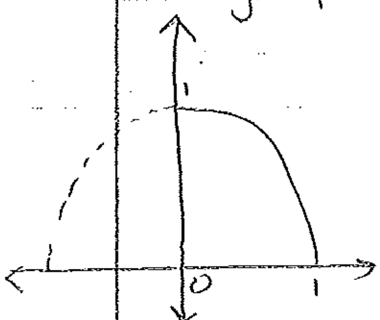


$$X_{cm} = \frac{3+6+0}{3} = \frac{3+6}{3}$$

$$Y_{cm} = \frac{0+0+6}{3} = 2$$

$$COM = \left(\tfrac{3+6}{3},\ 2\right)$$

4. Find the centroid of the region under the graph of $y = 1-x^2$ for $0 \le x \le 1$.



$$M_y = \rho\int_0^1 x(1-x^2)\,dx$$
$$= \rho\int_0^1 x - x^3\,dx$$
$$= \rho\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1$$
$$= \rho\left[\left(\tfrac{1}{2} - \tfrac{1}{4}\right)\right]$$
$$= \tfrac{1}{4}\rho$$

$y = 1-u$

let $u = 1-y$

$du = -dy$

$x = \sqrt{1-y}$

$$M_y = \rho\int_0^1 y\sqrt{1-y}\,dy$$
$$= -\rho\int_0^1 (1-u)u^{1/2}\,du$$
$$= -\rho\int_0^1 u^{1/2} - u^{3/2}\,du$$
$$= -\rho\left(\tfrac{2}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right)\Big|_0^1$$
$$= -\rho\left(\tfrac{2}{3} - \tfrac{2}{5}\right)$$

$$M_x = \frac{\rho}{2}\int_0^1 1 - 2x^2 + x^4\,dx$$
$$= \tfrac{\rho}{2}\left(x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_0^1$$
$$= \tfrac{\rho}{2}\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)$$
$$= \tfrac{\rho}{2}\left(\tfrac{1}{3} + \tfrac{1}{5}\right)$$
$$= \tfrac{4\rho}{15}$$

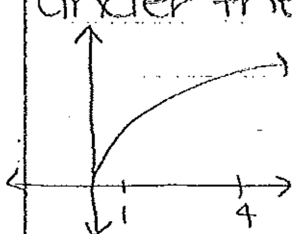
$$M = \rho\int_0^1 1 - x^2\,dx$$
$$= \rho\left(x - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= \rho\left(1 - \tfrac{1}{3}\right)$$
$$= \tfrac{2}{3}\rho$$

$$X_{cm} = \frac{\rho/4}{2\rho/3}$$
$$= \frac{\rho}{4}\cdot\frac{3}{2\rho}$$
$$= \tfrac{3}{8}$$

$$Y_{cm} = \frac{\overset{2}{\cancel{4}}\cancel{\rho}}{\underset{5}{\cancel{15}}}\cdot\frac{\cancel{3}}{\cancel{2}\cancel{\rho}} = \tfrac{2}{5}$$

$$\boxed{\left(\tfrac{3}{8},\ \tfrac{2}{5}\right)}$$

5. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \le x \le 4$.



$$M_y = \rho\int_1^4 x(x^{1/2})\,dx$$
$$= \rho\int_1^4 x^{3/2}\,dx$$
$$= \rho\left(\tfrac{2}{5}x^{5/2}\right)\Big|_1^4$$
$$= \tfrac{2}{5}\rho(32-1) = \tfrac{62}{5}\rho$$

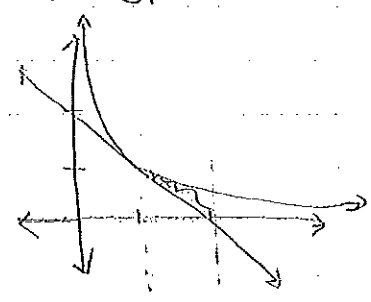
$$M_x = \frac{\rho}{2}\int_1^4 x\,dx$$
$$= \frac{\rho}{2}\left(\frac{1}{2}x^2\right)\Big|_1^4$$
$$= \rho/2\left(8 - \frac{1}{2}\right)$$
$$= \rho/2\,(15/2)$$
$$= 15\rho/4$$

$$M = \rho\int_1^4 \sqrt{x}\,dx$$
$$= \rho\left(\frac{2}{3}x^{3/2}\right)\Big|_1^4$$
$$= \rho\,(16/3 - 2/3)$$
$$= 14\rho/3$$

$$x_{CM} = \frac{\overset{31}{\cancel{62}}\cancel{\rho}}{5}\cdot\frac{3}{\cancel{14}\cancel{\rho}} = \frac{93}{35} \qquad y_{CM} = \frac{15\cancel{\rho}}{4}\cdot\frac{3}{14\cancel{\rho}} = \frac{45}{56}$$

$$\boxed{(93/35,\ 45/56)}$$

* 6. Find the centroid of the region between $f(x) = x^{-1}$ and $g(x) = 2-x$ for $1 \le x \le 2$.



$$M_y = \rho\int_1^2 x\,(x^{-1} - 2 + x)\,dx$$
$$= \rho\int_1^2 1 - 2x + x^2\,dx$$
$$= \rho\,(x - x^2 + 1/3\,x^3)\Big|_1^2$$
$$= \rho\,((2 - 4 + 8/3) - (1/3))$$
$$= \rho\,(-6/3 + 8/3 - 1/3)$$
$$= 1/3\,\rho$$

$$M_x = \frac{\rho}{2}\int_1^2 x^{-2} - 4 + 2x - x^2\,dx$$
$$= \frac{\rho}{2}\left(-x^{-1} - 4x + x^2 - 1/3\,x^3\right)\Big|_1^2$$
$$= \frac{\rho}{2}\left((-1/2 - 8 + 4 - 8/3) - (-1 - 4 + 1 - 1/3)\right)$$
$$= \rho/2\,(-3/6 - 16/6 + 2/6)$$
$$= -1/12\,\rho$$

$$x_{CM} = \frac{\cancel{\rho}}{3}\cdot\frac{1}{\cancel{\rho}(\ln|2| - 1/2)}$$
$$= 1/(3\ln|2| - 3/2)$$

$$M = \rho\int_1^2 x^{-1} - 2 + x\,dx$$
$$= \rho\left(\ln|x| - 2x + \frac{1}{2}x^2\right)\Big|_1^2$$
$$= \rho\,(\ln|2| - \cancel{4} + \cancel{2} - 0 + \cancel{2} - 1/2)$$
$$= \rho\,(\ln|2| - 1/2)$$

$$y_{CM} = \frac{-17\cancel{\rho}}{12}\cdot\frac{1}{\cancel{\rho}(\ln|2| - 1/2)}$$
$$= 1/(12\ln|2| - 6)$$

7. Let $m > n \geq 0$. Find the centroid of the region between x^m and x^n for $0 \leq x \leq 1$. Find values for m, n that force the centroid to lie outside of the region.

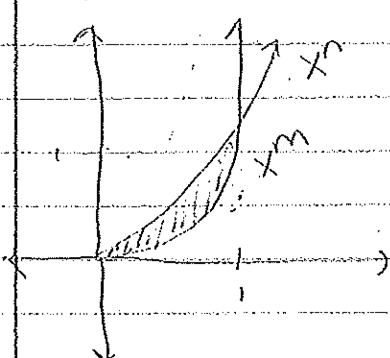
$$M_y = \rho \int_0^1 x(x^n - x^m)\,dx$$
$$= \rho \int_0^1 x^{n+1} - x^{m+1}\,dx$$
$$= \rho\left(\frac{1}{n+2}x^{n+2} - \frac{1}{m+2}x^{m+2}\right)\Big|_0^1$$
$$= \rho\left(\frac{1}{n+2} - \frac{1}{m+2}\right)$$
$$\frac{m-2 \quad - n+2}{(n+2)(m+2)}$$

$$M_x = \frac{\rho}{2}\int_0^1 x^{2n} - x^{2m}\,dx$$
$$= \frac{\rho}{2}\left(\frac{1}{2n+1}x^{2n+1} - \frac{1}{2m+1}x^{2m+1}\right)\Big|_0^1$$
$$= \frac{\rho}{2}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right)$$

$$M = \rho\int_0^1 x^n - x^m\,dx$$
$$= \rho\left(\frac{1}{n+1}x^{n+1} - \frac{1}{m+1}x^{m+1}\right)\Big|_0^1$$
$$= \rho\left(\frac{1}{m+1} - \frac{1}{n+1}\right)$$

$$x_{CM} = \frac{\cancel{\rho}\left(\frac{1}{n+2} - \frac{1}{m+2}\right)}{\cancel{\rho}\left(\frac{1}{n+1} - \frac{1}{m+1}\right)}$$
$$= \frac{\cancel{n-m}}{(m+2)(n+2)} \cdot \frac{(m+1)(n+1)}{\cancel{n-m}}$$
$$\boxed{= \frac{(m+1)(n+1)}{(m+2)(n+2)}}$$

$$y_{CM} = \frac{\frac{\cancel{\rho}}{2}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right)}{\cancel{\rho}\left(\frac{1}{n+1} - \frac{1}{m+1}\right)}$$
$$= \frac{2n - 2\cancel{m}}{2(2n+1)(2m+1)} \cdot \frac{(m+1)(n+1)}{\cancel{n-m}}$$
$$\boxed{= \frac{(m+1)(n+1)}{(2m+1)(2n+1)}}$$



for large m, n.

2. $M = 4+2+5+1 = 12$

$M_x = 4\cdot 2 + 2\cdot 2 + 5\cdot(-1) + 0\cdot 1 = 8+4+-5+0 = 7$

$M_y = 4\cdot 1 + 2(-3) + 5\cdot 2 + 1\cdot 4 = 4 + -6 + 10 + 4 = 12$

So, $x_{cm} = \frac{12}{12} = 1 \quad y_{cm} = \frac{7}{12}$

So $COM = (1, \frac{7}{12})$

4) (Alternate solution for M_y)

$$M_y = \frac{1}{2}\rho \int_a^b f(y)^2\,dy \qquad y = 1-x^2$$

$$x^2 = 1-y$$

$$x = \sqrt{1-y}$$

$$= \frac{1}{2}\rho\int_0^1 (\sqrt{1-y})^2\,dy$$

$$= \frac{1}{2}\rho\int_0^1 (1-y)\,dy = \frac{1}{2}\rho\left[y - \frac{y^2}{2}\right]\Big|_0^1$$

$$= \frac{1}{2}\rho\left[1 - \frac{1}{2}\right] = \frac{1}{4}\rho$$

$$M_x = \rho\int_a^b y\, f(y)\,dy \qquad u = 1-y$$

$$du = -dy$$

$$= \rho\int_0^1 y\sqrt{1-y}\,dy$$

$$= \rho\int_1^0 -(1-u)\sqrt{u}\,du$$

$$= \rho\int_0^1 \sqrt{u} - u^{3/2}\,du$$

$$= \rho\left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1 = \rho\left(\frac{2}{3} - \frac{2}{5}\right)$$

$$= \frac{4}{15}\rho$$

8. $S = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$

$2\pi \int_0^4 x\sqrt{1+1^2}\,dx = 2\pi \int_0^4 x\sqrt{2}\,dx$

$= 2\sqrt{2}\,\pi\left[\left(\frac{1}{2}x^2\right)\right]\Big|_0^4$

$= 2\sqrt{2}\,\pi\left[\frac{1}{2}\cdot 16 - \frac{1}{2}(0)\right]$

$= 2\sqrt{2}\,\pi(8-0)$

$= 16\sqrt{2}\,\pi$

9. $S = 2\pi \int_0^2 x^3\sqrt{1+(3x^2)^2}\,dx$

$= 2\pi \int_0^2 x^3\sqrt{1+9x^4}\,dx$ Let $u = 1+9x^4$

$du = 36x^3\,dx$

$= \frac{2\pi}{36}\int_1^{145}\sqrt{u}\,du$

$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145} = \frac{\pi}{27}\left(145^{3/2} - 1\right) \approx 203.0436$

10) $f(x) = (4 - x^{2/3})^{3/2}$

$f'(x) = 3/2\,(4-x^{2/3})^{1/2} \cdot -2/3\,x^{-1/3} = -x^{-1/3}\sqrt{4-x^{2/3}}$

$S = 2\pi\int_0^8 (4-x^{2/3})^{3/2}\sqrt{1+\left(-x^{-1/3}\sqrt{4-x^{2/3}}\right)^2}\,dx$

$= 2\pi\int_0^8 (4-x^{2/3})^{3/2}\sqrt{1+x^{-2/3}(4-x^{2/3})}\,dx$

$= \frac{384\pi}{5} \approx 241.274$