MA 114 Worksheet 22 - Calculus with Parametric Curves

1) a) $x = e^{\sqrt{t}}$

$\frac{dx}{dt} = e^{\sqrt{t}} \cdot \frac{1}{2}t^{-1/2} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$ $\quad x'(1) = \frac{e^{\sqrt{1}}}{2\sqrt{1}} = \frac{e}{2}$ $\quad x(1) = e$

$y = t - \ln t^2$

$\frac{dy}{dt} = 1 - \frac{1}{t^2} \cdot 2t = 1 - \frac{2}{t}$ $\quad y'(1) = 1 - \frac{2}{1} = -1$ $\quad y(1) = 1$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y'(1)}{x'(1)} = \frac{-1}{e/2} = -2/e$ } slope of tangent line

$(e, 1)$ is a point on the tangent line when $t = 1$.

$$\boxed{y - 1 = -\frac{2}{e}(x - e)}$$

b) $x = \cos\theta + \sin 2\theta$

$\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta \cdot 2 = 2\cos 2\theta - \sin\theta$ $\quad x'(\pi/2) = 2\cos(\pi) - \sin(\pi/2)$
$= -2 - 1 = -3$

$y = \cos\theta$ $\quad y'(\pi/2) = -\sin(\pi/2)$ $\quad x(\frac{\pi}{2}) = 0$

$\frac{dy}{d\theta} = -\sin\theta$ $\quad = -1$ $\quad y(\frac{\pi}{2}) = 0$

$\frac{dy}{dx} = \frac{-1}{-3} = 1/3$ } slope of tangent line

$(0, 0)$ is a point on the tangent line when $\theta = \frac{\pi}{2}$.

$$\boxed{y = \frac{1}{3}x}$$

2a) $x = t - e^t$ $y = t + e^{-t}$

$x' = 1 - e^t$ $y = 1 + -e^{-t}$

$\frac{dy}{dx} = \frac{1 - e^{-t}}{1 - e^t}$

b) $x = t^3 - 12t$ $y = t^2 - t$

$x' = 3t^2 - 12$ $y' = 2t - 1$

$\frac{dy}{dx} = \frac{2t-1}{3t^2-12}$

c) $x = 2\cos(2t)$ $y = \cos(t)$

$x' = 2\sin(2t) \cdot 2$ $y' = -\sin(t)$

$x' = -4\sin(2t)$

$\frac{dy}{dx} = \frac{-\sin(t)}{-4\sin(2t)} = \frac{\sin(t)}{4\sin(2t)}$

$A = \int_\alpha^\beta y(t)x'(t)dt$ by symmetry:

p. 619 #89 5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $A = 2\int_0^\pi y(t)x'(t)dt$

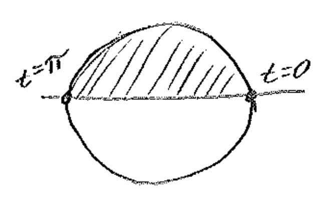
$c(t) = (a\cos(t), b\sin(t))$ for $0 \le t < 2\pi$

$A = 2\int_\pi^0 -b\sin(t)\, a\sin(t)\,dt$

$2\int_0^\pi ab \sin^2 t \, dt$

$2\int_0^\pi ab \left(\frac{1-\cos(2t)}{2}\right)dt$

$ab\int_0^\pi (1-\cos(2t))dt$



$ab\left[t - \frac{\sin 2t}{2}\right]_0^\pi$

$ab[\pi - 0] = ab\pi$

4) Let $x = 8t^2+9$ and $y = 1-4t$

$$\frac{dx}{dt} = 16t \qquad \frac{dy}{dt} = -4$$

$$\frac{d^2x}{dt^2} = 16 \qquad \frac{d^2y}{dt^2} = 0$$

$$\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{0}{16} = 0$$

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3}$$

$$= \frac{16t(0) - (-4)(16)}{(16t)^3} = \frac{64}{4096t^3} = \frac{1}{64t^3}$$

$$0 \neq 1/64t^3$$

So $\dfrac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} \neq \dfrac{d^2y}{dx^2}$

3. $x = 7 + t^2 + e^t$, $y = \cos(t) + \frac{1}{t}$ $\quad 0 < t \le \pi$.

$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} \quad \text{(page 618)}$$

$x'(t) = 2t + e^t \qquad x''(t) = 2 + e^t$

$y'(t) = -\sin(t) - 1/t^2 \qquad y''(t) = -\cos(t) + 2/t^3$

so $$\frac{d^2y}{dx^2} = \frac{(2t + e^t)(-\cos(t) + 2/t^3) - (-\sin(t) - 1/t^2)(2 + e^t)}{(2t + e^t)^3}$$

6. Arc length is given by $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$$= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= 6 \int_0^1 \sqrt{t^2 + t^4}\, dt$$

$$= 6 \int_0^1 t\sqrt{1+t^2}\, dt \qquad \text{Let } u = 1+t^2$$

$$du = 2t\,dt$$

$$= 3 \int_1^2 \sqrt{u}\, du$$

$$= 3 \left[\tfrac{2}{3} u^{3/2}\right]_1^2$$

$$= 2\left(2^{3/2} - 1\right)$$