Worksheet 23 Key

1. a) $\dfrac{x^2+1}{x^2+3x+2} = \dfrac{x^2+1}{(x+2)(x+1)} = 1 + \dfrac{A}{x+2} + \dfrac{B}{x+1}$ (Use long division first.)

b) $\dfrac{x+2}{(x^2+1)^2(x+5)(x+3)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2} + \dfrac{E}{x+5} + \dfrac{F}{x+3}$

c) $\dfrac{x+2}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2}$

$x+2 = A(x-2) + B(x+5)$

Try $x=2$: $4 = B \cdot 7 \Rightarrow B = 4/7$

Try $x=-5$: $-3 = -7A \Rightarrow A = 3/7$

$$\int \frac{x+2}{(x+5)(x-2)}\,dx = \frac{3}{7}\int \frac{1}{x+5}\,dx + \frac{4}{7}\int \frac{1}{x-2}\,dx$$

$$= \frac{3}{7}\ln|x+5| + \frac{4}{7}\ln|x-2| + C$$

2. a) $\displaystyle\lim_{R\to\infty} \int_0^R \frac{x}{(x^2+1)^2}\,dx$ Let $u = x^2+1$, $du = 2x\,dx$

$$= \lim_{R\to\infty} \frac{1}{2}\int_1^{R^2+1} \frac{1}{u^2}\,du = \lim_{R\to\infty} \frac{1}{2}\left[-\frac{1}{u}\right]_1^{R^2+1}$$

$$= \lim_{R\to\infty} \frac{1}{2}\left[-\frac{1}{R^2+1} + 1\right] = \frac{1}{2}$$

b) $\displaystyle\lim_{R\to\infty} \int_{-R}^0 \frac{x}{x^2+1}\,dx + \lim_{R\to\infty}\int_0^R \frac{x}{x^2+1}\,dx$ $u = x^2+1$, $du = 2x\,dx$

$$= \lim_{R\to\infty} \int_{R^2+1}^1 \frac{1}{2u}\,du + \lim_{R\to\infty}\int_1^{R^2+1} \frac{1}{2u}\,du$$

$$= \lim_{R\to\infty} \frac{1}{2}\left[0 - \ln|R^2+1|\right] + \lim_{R\to\infty} \frac{1}{2}\left[\ln|R^2+1| - 0\right]$$

diverges

1. c) $\lim_{R\to 1} \int_0^R \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{R\to 1} \left[\sin^{-1}(x)\right]_0^R$

$= \lim_{R\to 1} \sin^{-1}(R) - \sin^{-1}(0)$

$= \pi/2$

d) $\lim_{R\to\infty} \int_e^R \frac{1}{x(\ln(x))^3}\,dx$ Let $u = \ln(x)$, $du = \frac{1}{x}\,dx$

$= \lim_{R\to\infty} \int_1^{\ln(R)} \frac{1}{u^3}\,du$

$= \lim_{R\to\infty} \left[-\frac{1}{2u^2}\right]_1^{\ln(R)}$

$= \lim_{R\to\infty} -\frac{1}{2}\left[\frac{1}{\ln(R)^2} - 1\right] = \frac{1}{2}$

3. On $[1,\infty)$, the integrand is nonnegative. So look at the absolute value:

$\left|\frac{2+\sin(x)}{x^3+x}\right| \le \frac{3}{x^3+x} \le \frac{3}{x^3}$, and

$\int_1^\infty \frac{3}{x^3}\,dx$ converges, so the original integral converges as well by comparison.

MA 114 Worksheet 23 Review for Exam III

4) $\int_1^\infty \frac{1}{n^2+1}$

Examine

$$\int_1^R \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_1^R = \arctan(R) - \arctan(1)$$

$$= \arctan(R) - \pi/4$$

$$\lim_{R\to\infty} \int_1^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty}\left(\arctan(R) - \pi/4\right) = \pi/2 - \pi/4 = \pi/4$$

So $\int_1^\infty \frac{1}{n^2+1}$ converges and thus $\sum_{n=1}^\infty \frac{1}{n^2+1}$ also converges

use $\int_{M+1}^\infty f(x) \le S - \sum_{n=1}^{M} a_n \le a_{M+1} + \int_{M+1}^\infty f(x)$

(from page 568)

$$\lim_{R\to\infty} \int_{12}^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \left(\arctan(R) - \arctan(11)\right) = \pi/2 - \arctan(11)$$

$$a_{n+1} = a_{10+1} = a_{11} = \frac{1}{11^2+1} = \frac{1}{122}$$

$$\lim_{R\to\infty} \int_{12}^R \frac{1}{x^2+1}\,dx = \pi/2 - \arctan(11)$$

Smallest possible error: $\pi/2 - \arctan(11)$

Largest possible error: $\frac{1}{122} + \frac{\pi}{2} - \arctan(11)$

5) a) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \quad , [1, 2e]$

$y' = \frac{1}{2}x - \frac{1}{2x}$

$$S = \int_1^{2e} \sqrt{1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2}\, dx = \int_1^{2e} \sqrt{1 + \left(\frac{1}{4}x^2 + \frac{1}{4x^2} - \frac{1}{2}\right)}\, dx$$

$$= \int_1^{2e} \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{4x^2} + \frac{1}{2}\right)}\, dx = \int_1^{2e} \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2}\, dx = \int_1^{2e} \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\ln|2x| \Big|_1^{2e} = \frac{(2e)^2}{4} + \frac{1}{2}\ln|4e| - \left(\frac{1}{4} + \frac{1}{2}\ln|2|\right)$$

$$= e^2 + \frac{1}{2}\ln|4e| - \frac{1}{4} - \frac{1}{2}\ln|2|$$

b) $y = \sin(x), \; [0, \pi]$ rotated about x-axis

$y' = \cos(x)$

$$S = 2\pi \int_0^{\pi} \sin x \sqrt{1 + \cos^2 x}\, dx$$

$u = \cos x$
$du = -\sin x\, dx$
$-du = \sin x\, dx$

$$= -2\pi \int_1^{-1} \sqrt{1+u^2}\, du = 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$

$$= 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta = 2\pi \int_{-\pi/4}^{\pi/4} \sqrt{\sec^2\theta} \cdot \sec^2\theta\, d\theta$$

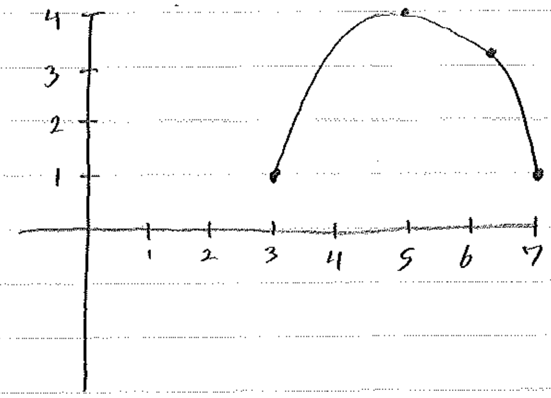
$$= 2\pi \int_{-\pi/4}^{\pi/4} \sec^3\theta\, d\theta$$

* see worksheet 15 for this integration

6. a) $x(t) = 2\cos t + 5$ and $y(t) = 3\sin t + 1$ for $0 \le t \le \pi$

Sketch the path a[illegible] direction.

t	$x(t)$	$y(t)$
0	$2+5=7$	$0+1=1$
$\pi/4$	$\sqrt{2}+5$	$\frac{3\sqrt{2}}{2}+1$
$\pi/2$	$0+5=5$	$3+1=4$
π	$-2+5=3$	$0+1=1$



(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\quad x(t) = t + e^t \quad y(t) = e^{-t}$

$x'(t) = 1 + e^t \quad y'(t) = -e^{-t}$

$$\frac{dy}{dx} = \frac{-e^{-t}}{1+e^t}$$

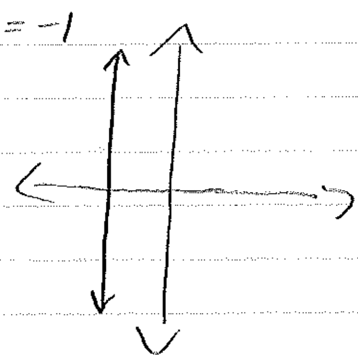
$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^3} \qquad y''(t) = e^{-t} \quad x''(t) = e^t$$

$$= \frac{(1+e^t)(e^{-t}) - (-e^{-t})(e^t)}{(1+e^t)^3} = \frac{e^{-t} + e^{t^2}}{(1+e^t)^3}$$

7. (a) Graph polar equations

$r = -\sec\theta$

$x = r\cos\theta = -1$



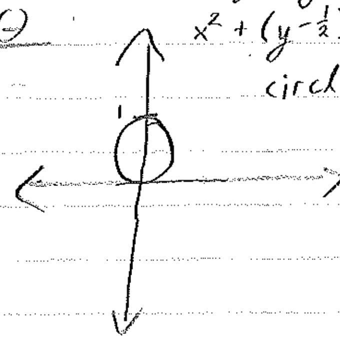
θ	$\sin\theta$
0	0
$\pi/2$	1
π	0
$3\pi/2$	-1
2π	0

$r^2 = r\sin\theta$

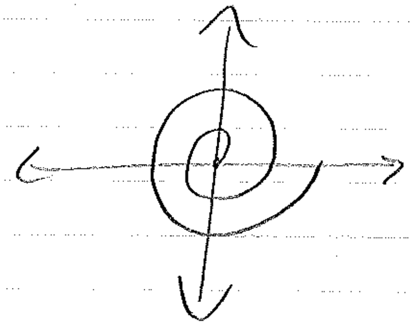
$x^2 + y^2 = y$

$x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$

circle!



r	θ
0	0
$\pi/2$	$\pi/2$
π	π
2π	2π



(b) $x^2+(y-1)^2=1$ to polar coordinates.

$x^2+(y-a)^2=a^2 \Rightarrow r=2a\sin\theta$

$r=2(1)\sin\theta$

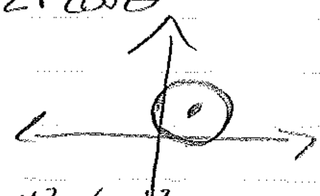
(c) Convert $r=2\sin\theta+2\cos\theta$ to rectangular and graph.

$r^2=2r\sin\theta+2r\cos\theta$

$r=\sqrt{x^2+y^2}$

$r^2=2y+2x$

$x^2+y^2=2y+2x$



$x^2+y^2-2y-2x=0$

$x^2-2x+1+y^2-2y+1=2 \Rightarrow (x-1)^2+(y-1)^2=2$

(d) Find slope of tangent line to polar curve $r=\theta$ at $\pi/2$

$x=\theta\cos\theta \qquad y=\theta\sin\theta$

$x'=1(\cos\theta)-\theta\sin\theta \qquad y'=1\sin\theta+\theta\cos\theta$

$$\frac{dy}{dx}=\frac{\sin\theta+\theta\cos\theta}{\cos\theta-\theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2}=\frac{1}{-\pi/2}=-\frac{2}{\pi}$$