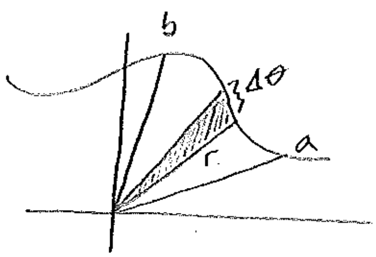
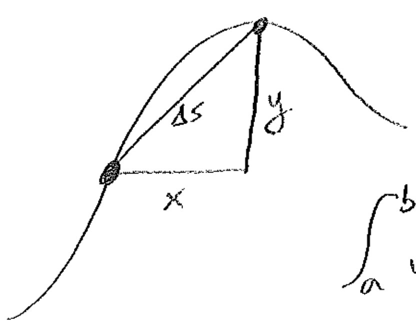
MA 114 Worksheet 25 - Calculus w/ Polar Coordinates

1) a) Area $= \frac{1}{2}\int_a^b r^2\,d\theta = \frac{1}{2}\int_a^b f(\theta)^2\,d\theta$



The area of a slice is calculated approximately as the area of a circular sector of radius r : $\frac{1}{2} r^2 \Delta\theta$. The limit of the Riemann sum gives the integral.

b) Arc length $= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$



We have $r = f(\theta) = \frac{x}{\cos(\theta)} = \frac{y}{\sin(\theta)}$.

So $x = f(\theta)\cos(\theta)$, $y = f(\theta)\sin\theta$.

Then recall arc length is given by

$$\int_a^b \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta = \ldots = \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta.$$

c) A circle of (constant) radius r has area

$$\frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\left[\theta r^2\right]_0^{2\pi} = \pi r^2$$

and circumference (thinking $f(\theta) = r$, so $f'(\theta) = 0$)

$$\int_0^{2\pi} \sqrt{r^2 + 0^2}\,d\theta = \int_0^{2\pi} r\,d\theta = \left[r\theta\right]_0^{2\pi} = 2\pi r$$

5. $r = \cos(\theta)$, $0 \le \theta \le \pi/4$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} \cos^2(\theta)\, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1}{2} + \frac{1}{2}\cos(2x)\, dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}\, d\theta + \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}\cos(2x)\, dx$$

$$= \frac{1}{4}\left(\int_0^{\pi/4} 1\, d\theta + \int_0^{\pi/4} \cos(2x)\, dx \right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{1}{2}\Big[\sin(2x)\Big]_0^{\pi/4} \right)$$

$$= \frac{\pi}{16} + \frac{1}{8}$$

2) $r=\theta^2$

$x = r\cos\theta = \theta^2\cos\theta$

$x' = -\theta^2\sin\theta + 2\theta\cos\theta \qquad x'(\pi) = -\pi^2(0) - 2\pi = -2\pi$

$y = r\sin\theta = \theta^2\sin\theta$

$y' = \theta^2\cos\theta + 2\theta\sin\theta \qquad y'(\pi) = \pi^2(-1) + 2(\pi)(0) = -\pi^2$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{y'(\theta)}{x'(\theta)} = \frac{y'(\pi)}{x'(\pi)} = \frac{-\cancel{\pi}^2}{-2\cancel{\pi}} = \frac{\pi}{2}$$

3) $r = 2+\sin\theta$

$x = r\cos\theta = (2+\sin\theta)(\cos\theta) = 2\cos\theta + \sin\theta\cos\theta$

$x' = -2\sin\theta - \sin^2\theta + \cos^2\theta$

$y = r\sin\theta = (2+\sin\theta)\sin\theta = 2\sin\theta + \sin^2\theta$

$y' = 2\cos\theta + 2\sin\theta\cos\theta$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{2\cos\theta + 2\sin\theta\cos\theta}{-2\sin\theta - \sin^2\theta + \cos^2\theta} = \frac{2\cos\theta\,(1+\sin\theta)}{\cos^2\theta - 2\sin\theta - \sin^2\theta}$$

$\frac{dy}{dx} = 0$ when $\cos\theta = 0$ or $\sin\theta = -1$, thus

when $\theta = \frac{\pi}{2} + 2\pi n$ or $\theta = \frac{3\pi}{2} + 2\pi n$.

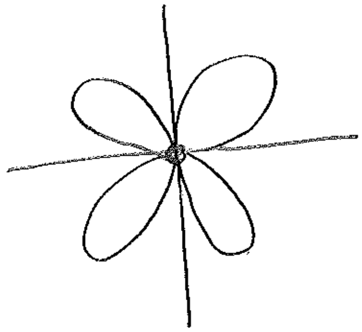
(Or more simply, $\theta = \frac{\pi}{2} + \pi n$.)

One can quickly verify that $\cos^2(\frac{\pi}{2}) - 2\sin(\frac{\pi}{2}) - \sin^2(\frac{\pi}{2}) = -3 \neq 0$

and $\cos^2(\frac{3\pi}{2}) - 2\sin(\frac{3\pi}{2}) - \sin(\frac{3\pi}{2}) = 1 \neq 0$.

4) $r = \sin 2\theta$

θ	r
0	0
$\pi/2$	0



$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (\sin 2\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{1}{2} - \frac{\cos(4\theta)}{2}\right]d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{1}{2} - \frac{1}{2}\int_0^{\pi/2}\frac{\cos(4\theta)}{2}\,d\theta$$

$u = 4\theta$
$du = 4\,d\theta$
$\frac{du}{4} = d\theta$

$$= \frac{1}{2}\left(\frac{\theta}{2}\right)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{2\pi}\frac{\cos u}{8}\,du$$

$$= \frac{1}{2}\left(\frac{\theta}{2}\right)\Big|_0^{\pi/2} - \frac{1}{2}\left(\frac{\sin u}{8}\right)\Big|_0^{2\pi}$$

$$= \frac{\theta}{4}\Big|_0^{\pi/2} - \frac{\sin u}{16}\Big|_0^{2\pi}$$

$$= \frac{\pi}{8}$$