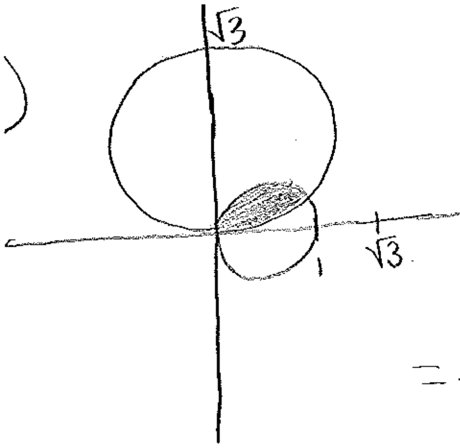
MA 114 Worksheet 26 – Calculus w/ Polar Coordinates II and Differential Equations

1) a)



$$A = \frac{1}{2}\int_0^{\pi/6} (\sqrt{3}\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (\cos\theta)^2 d\theta$$

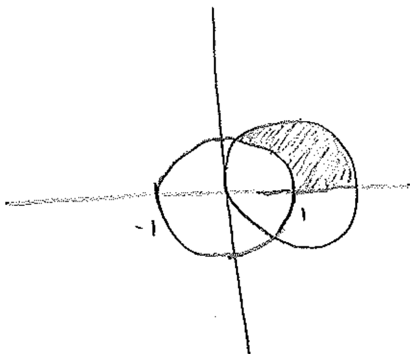
$$= \frac{1}{2}\int_0^{\pi/6} 3\sin^2\theta \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} 3\left(\frac{1-\cos 2\theta}{2}\right) d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \left(\frac{3}{2} - \frac{3\cos 2\theta}{2}\right) d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - \frac{3\sin 2\theta}{4}\right]_0^{\pi/6} + \frac{1}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/6}^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] + \frac{\pi}{8} - \frac{1}{2}\left[\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right]$$

b)



$$A = \frac{1}{2}\int_0^{\pi/3} (4\cos^2\theta - 1) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \left(\frac{4+4\cos 2\theta}{2} - 1\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} (1 + 2\cos 2\theta) d\theta \quad \frac{1}{2}\left[\theta + \frac{2\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]_0^{\pi/3}$$

$$= \frac{1}{2}\left[\pi/3 + \frac{\sqrt{3}}{2}\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

$1 = 2\cos\theta$

$1/2 = \cos\theta$

$\theta = \arccos(1/2)$
$= \pi/3$

c) $r=\theta^2$

$r'=2\theta$

$$S=\int_0^{2\pi}\sqrt{(\theta^2)^2+(2\theta)^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta=\int_0^{2\pi}\sqrt{\theta^2(\theta^2+4)}\,d\theta$$

$$=\int_0^{2\pi}\theta\sqrt{\theta^2+4}\,d\theta$$

$u=\theta^2+4$

$du=2\theta d\theta$

$\frac{du}{2}=\theta d\theta$

$$=\int_4^{4\pi^2+4}\frac{1}{2}\sqrt{u}\,du=\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_4^{4\pi^2+4}=\frac{1}{3}(\theta^2+4)^{3/2}\Big|_0^{2\pi}$$

$$=\frac{1}{3}(4\pi^2+4)^{3/2}-\frac{1}{3}(4)^{3/2}$$

d) $r=\sin\theta+\theta$

$r'=\cos\theta+1$

$$S=\int_0^{\pi}\sqrt{(\sin\theta+\theta)^2+(\cos\theta+1)^2}\,d\theta$$

2. a) $y = \sin(3x) + 2e^{4x}$

$y' = 3\cos(3x) + 8e^{4x}$

$y'' = -9\sin(3x) + 32e^{4x}$

Check the DE: $y'' + 9y = 50e^{4x}$

$$-9\sin(3x) + 32e^{4x} + 9(\sin(3x) + 2e^{4x}) \stackrel{?}{=} 50e^{4x}$$

$$32e^{4x} + 18e^{4x} = 50e^{4x} \checkmark$$

So y is a solution to the DE.

b) Since $y' = y^2 + 6$ is always positive, any solution function y must be always increasing.

c) A differential equation is <u>linear</u> if it is of the form $a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$, and is <u>nonlinear</u> otherwise.

3. $y = e^{rx}$ $\quad y' = re^{rx}$ $\quad y'' = r^2e^{rx}$

$$y'' + y' - 12y = 0$$

$$r^2e^{rx} + re^{rx} - 12e^{rx} = 0$$

$$r^2 + r - 12 = 0$$

$$(r+4)(r-3) = 0$$

$$r = -4, 3$$

4(a) $y' + 4xy^2 = 0$

$4xy^2 = -y'$

$4x = \frac{-1}{y^2} dy/dx$

$4x\,dx = \frac{-1}{y^2} dy$

$\int 4x\,dx = \int \frac{-1}{y^2}\,dy$

$2x^2 + C = \frac{1}{y}$

$y = \frac{1}{2x^2+c}$

(b) $\sqrt{1-x^2}\,\frac{dy}{dx} = x^3 y$

$\int \frac{1}{y}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx$ Let $u = 1-x^2$

$\ln|y| = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$ $du = -2x\,dx$

$\ln|y| = -\frac{1}{2}(2\sqrt{1-x^2}) + c$ $-\frac{1}{2}du = x\,dx$

$\ln|y| = -\sqrt{1-x^2} + c$

$y = e^{-\sqrt{1-x^2}+c}$

(c) $(1+x^2)\frac{dy}{dx} = x^3 y$

$\int \frac{1}{y}\,dy = \int \frac{x^3}{1+x^2}\,dx$

$$\begin{array}{r} x \\ x^2+1\,\overline{)\,x^3} \\ \underline{x^3+x} \\ -x \end{array}$$

$\ln|y| = \int x + \frac{-x}{x^2+1}\,dx$

$= \int x - \frac{1}{2}\int \frac{1}{u+1}\,du$ $u = x^2$ $\frac{1}{2}du = x\,dx$

$\ln|y| = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C$

$y = e^{\frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + c}$

(d) $\sqrt{1+y^2}\,\frac{dy}{dx} + \sec x = 0$ $y = a\tan\theta$

$\int \sqrt{1+y^2}\,dy = \int -\sec x\,dx$ $dy =$

$= -\ln|\sec x + \tan x| + C$

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5. $\frac{dy}{dx} = 4y+24 \Rightarrow \frac{dy}{dx} = 4(y+6)$

$\Rightarrow \frac{1}{y+6}\,dy = 4\,dx \Rightarrow \int \frac{1}{y+6}\,dy = \int 4\,dx$

$$\ln|y+6| = 4x + K$$
$$y+6 = ce^{4x}$$
$$y = -6 + ce^{4x}$$

Use initial value $y(0)=5$: $5 = -6 + c$

$$c = 11$$

So $y = -6 + 11e^{4x}$ solves the DE.

6. $\frac{dy}{dx} = -6y+12 \Rightarrow \frac{dy}{dx} = -6(y-2)$

$\frac{1}{y-2}\,dy = -6\,dx = \int \frac{1}{y-2}\,dy = \int -6\,dx$

$$\ln|y-2| = -6x + K$$
$$y-2 = ce^{-6x}$$
$$y = 2 + ce^{-6x}$$

Use initial value $y(2)=10$: $10 = 2 + ce^{-6(2)}$

$$8 = ce^{-12}$$
$$c = 8e^{12}$$

So $y = 2 + 8e^{12-6x}$ solves the DE.