Worksheet 27 Key

1. $f(x) = y = x + 3e^{-x}$ $\qquad \frac{dy}{dx} = 1 - 3e^{-x}$

So we then have $\frac{dy}{dx} + y = 1 - 3e^{-x} + x + 3e^{-x}$
$= 1 + x.$

Thus $f(x)$ solves the DE.

2. a) Let $P(t)$ denote human population at time t.
By assumption, $\frac{dP}{dt} = K \cdot P(t)$ for some constant K.

Of course, this ignores many other factors which may contribute to population growth.

b) Let $V(t)$ denote volume of the raindrop at time t.
For a sphere of radius r, we have volume $V = \frac{4}{3}\pi r^3$ and area $A = 4\pi r^2$. Thus, solving for r, $r = \sqrt[3]{\frac{3V}{4\pi}}$. So $A = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$. Then by assumption, $\frac{dV}{dt} = K \cdot 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$ for some K.

Does this make sense?

c) Force due to gravity is given by mg, where m is mass of the object (in Kg) and g is acceleration due to gravity (in m/s^2). We assume force due to air resistance is Kv^2 for some constant K. Then total force of the system is

$$m \cdot v'(t) = mg - Kv^2.$$

4. $y' = -k(y - T_0)$, k = constant depending on the object, T_0 = temp. of the ambient environment

This uses 210° instead of 212°.

Frank's car engine runs at 210° F. On a 70° F day, he turns off the ignition and notes that five minutes later, the engine has cooled to 160° F.

(a) Find the cooling constant k.

$\int \frac{1}{y-70°}\,dy = \int -k\,dt$

$\ln|y-70°| = -kt + C$

$y - 70° = Ce^{-kt}$

$y = Ce^{-kt} + 70$

$210° = Ce^{0} + 70°$

$140 = C$

$160 = 140e^{-5k} + 70$

$90 = 140e^{-5k}$

$\frac{9}{14} = e^{-5k}$

$\ln|\frac{9}{14}| = -5k$

$\boxed{k = -\frac{1}{5}\ln(\frac{9}{14})}$

$\approx .08836$

(b) When will the engine cool to 100° F?

$100 = 140e^{\ln(9/14)^{-1/5t}} + 70$

$30 = 140(9/14)^{-t/5}$

$\ln(3/14) = -t/5(\ln(9/14))$

$\frac{\ln(3/14)}{\ln(9/14)} = -t/5$

$\frac{5\ln(3/14)}{\ln(9/14)} = t \approx 17.432$ should be 17.432

5. A cup of coffee with cooling constant $k = 0.09\,min^{-1}$ is placed in a room of temperature 20°C.

(a) How quickly is the coffee cooling when the temperature is 80°C?

$\frac{dy}{dt} = -k(y-20)$

$\frac{dy}{dt} = -0.09(80-20) = -0.09(60) = -5.4$

(b) Use the linear approximation to estimate the change in temperature over the next 6s when the temperature is 80°C.

$(\frac{1}{10})(-5.4) = -.54$

(c) If the coffee is initially served at 90°c, how long will it take to reach an optimal drinking temperature of 65°C.

$y = 20 + Ce^{-kt}$

$90 = 20 + Ce^{0}$

$70 = C$

$65 = 20 + 70e^{-.09t}$

$45 = 70e^{-.09t}$

$45/70 = e^{-.09t}$

$\ln|45/70| = -.09t$

$t = \frac{1}{-.09}\ln|45/70| \approx 4.9$ min.

6. $\frac{dy}{dt} = \frac{Bv(y)}{A(y)}$ $v(y) = -\sqrt{19.6y}$ $(= -\sqrt{2gy})$ $A(y) = \pi(\sqrt{y})^2 = \pi y$

(see p. 506, Rogawski)

$\frac{dy}{dt} = \frac{-0.0005\sqrt{19.6y}}{\pi y} \rightarrow \sqrt{y}\,dy = \frac{-0.0005\sqrt{19.6}}{\pi}\,dt$

$2/3\,y^{3/2} = \frac{-0.0005\sqrt{19.6}}{\pi}t + C$

At $y(0), 1$

$2/3 = C$

$2/3\,y^{3/2} = \frac{-0.0005\sqrt{19.6}}{\pi}t + 2/3$

Set $y=0$ to find when the tank is empty.

$0 = \frac{-0.0005\sqrt{19.6}}{\pi}t + 2/3$

$-2/3(\pi) = -0.0005\sqrt{19.6}\,t$

$t \approx 946.15$