Review Sheet

1. (a) $\int \frac{\sin(\ln(t))}{t}\,dt \qquad u=\ln(t) \quad du=\frac{1}{t}dt$

$\int \sin(u)\,du \quad = -\cos(\ln(t)) + C$

(b) $\int e^x \sin x\,dx \qquad u=e^x \quad dv=\sin x\,dx$

$du=e^x dx \quad v=-\cos x$

$-e^x\cos x + \int e^x \cos x\,dx \qquad u=e^x \quad dv=\cos x\,dx$

$du=e^x dx \quad v=\sin x$

$-e^x\cos x + e^x\sin x - \int e^x \sin x\,dx = \int e^x \sin x\,dx$

$$\boxed{\frac{-e^x\cos x + e^x \sin x}{2} + c}$$

(c) $\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \int_0^1 x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\Big|_0^1$

$= \frac{2}{3}(1)^{3/2} - 2(1)^{1/2} - 0 = \frac{2}{3} - \frac{6}{3} = -\frac{4}{3}$

(d) $\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \frac{x-1}{(x+2)(x+1)}\,dx$

Partial Fraction decomposition:

$\frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{(x+1)} = A(x+1) + B(x+2) \Rightarrow (A+B)x + (A+2B) = x-1$

$A+B=1 \quad A+2B=-1$

$A=1-B \quad (1-B)+2B=-1$

$1+1B=-1$

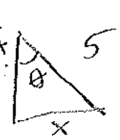
$B=-2$

$A=3$

$3\int_0^1 \frac{1}{x+2}\,dx + -2\int_0^1 \frac{1}{x+1}\,dx$

$= (3\ln(x+2) \quad -2\ln(x+1))\Big|_0^1 = 3\ln(3) - 2\ln(2) - 3\ln(2) + 2\ln(1)$

(e) $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$



$\sin\theta = \frac{x}{5} \qquad x = 5\sin\theta$

$5\cos\theta = \sqrt{25-x^2} \qquad dx = 5\cos\theta\,d\theta$

$= \int \frac{5\cos\theta}{25\sin^2\theta \cdot 5\cos\theta}\,d\theta = \frac{1}{25}\int \csc^2\theta\,d\theta = -\frac{1}{25}\cot\theta + c$

$= -\frac{\sqrt{25-x^2}}{25x} + c$

(f) $\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_t^0 xe^x dx$ $u = x$ $dv = e^x dx$
$du = 1dx$ $v = e^x$

$uv - \int v du = \lim_{t\to-\infty} \left(xe^x \Big|_t^0 - \int_t^0 e^x dx \right) = \lim_{t\to-\infty} \left(xe^x - e^x \right)\Big|_t^0$

$\lim_{t\to-\infty} \left(0e^0 - e^0 - te^t + e^t \right) = \lim_{t\to-\infty} -1 - te^t + e^t$

$= -1$ (converges)

5. (a) Solve $\frac{dL}{dt} = kL^2 \ln(t)$

$\int \frac{1}{L^2} dL = k \int \ln(t)\, dt$

$-\frac{1}{L} = K(t\ln(t) - t) + C$

$\frac{1}{L} = K(t - t\ln(t)) + c$

$L = \frac{1}{k(t - t\ln(t)) + c}$

Use $L(1) = -1$:

$-1 = \frac{1}{K(1 - 1 \cdot \ln(1)) + c}$

$-1 = \frac{1}{K + c}$

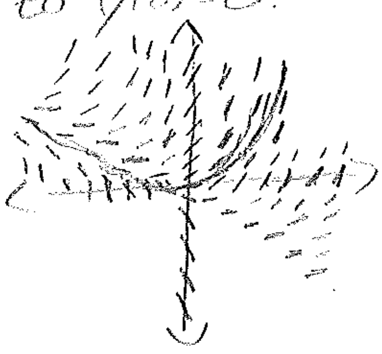
$K + c = -1$

$c = -1 - K$

Thus,

$L = \frac{1}{K(t - t\ln(t)) - 1 - K}$

(b) Draw direction field for $y' = y + x$. Sketch solution to $y(0) = 0$.



See Worksheet 29.

(c) Find solution of $y' - y = e^{2x}$ $y(0) = 1$.

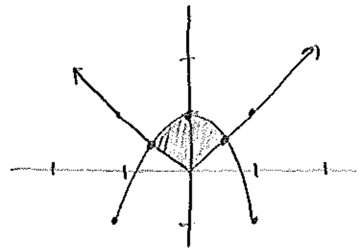
Guess: $y = e^{2x}$ since $y' = 2e^{2x}$ So, $y' - y = 2e^{2x} - e^{2x} = e^{2x}$

Also, $e^{2(0)} = 1$, so it satisfies $y(0) = 1$.

d) See Worksheet 29.

3. a) $y = 1-2x^2$

$y = |x|$



Solve for intersection:

$y = 1 - 2y^2$

$2y^2 + y - 1 = 0$

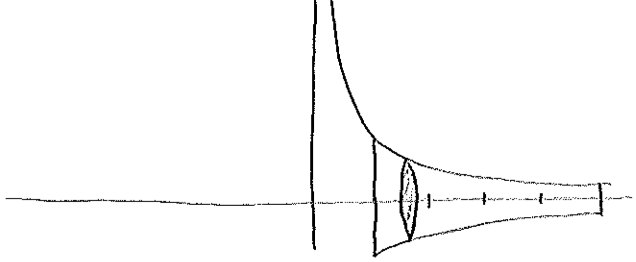
$(2y-1)\cancel{(y+1)} = 0$

$y = 1/2$

$x = \pm 1/2$

$$\text{Area} = \int_{-1/2}^{0} 1-2x^2-(-x)\,dx + \int_0^{1/2} 1-2x^2-x\,dx .$$

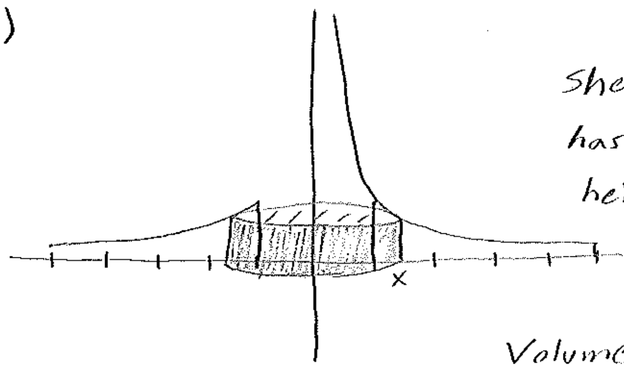
b)



Disc at x has radius $1/x$, so area π/x^2.

$$\text{Volume} = \int_1^5 \pi/x^2\,dx .$$

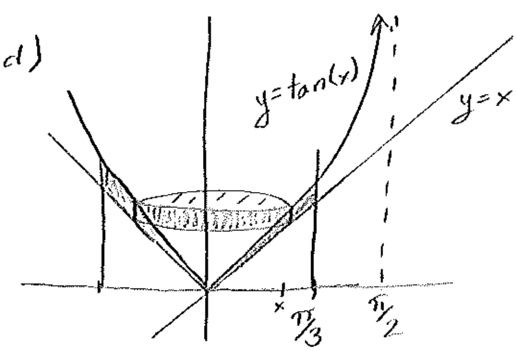
c)



Shell with right edge at x has radius x and height $1/x$, thus surface area $2\pi x \cdot 1/x = 2\pi$.

$$\text{Volume} = \int_1^5 2\pi\,dx .$$

d)



Shell with right edge at x has radius x and height $\tan(x)-x$, thus area $2\pi x(\tan(x)-x)$.

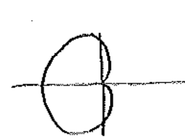
$$\text{Volume} = \int_0^{\pi/3} 2\pi x(\tan(x)-x)\,dx .$$

e) $f(x) = x^2$ from a to b.

Arc Length: $\int_a^b \sqrt{1+f'(x)^2}\,dx = \int_a^b \sqrt{1+(2x)^2}\,dx$

f) $x' = 6t$, $y' = 6t^2$, Arclength $= \int_0^2 \sqrt{(6t)^2+(6t^2)^2}\,dt$.

g) $r = 1-\cos\theta$ Area $= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\,d\theta$



MA 114 Worksheet 30 - Final Exam Review

4) a) $y = 2t \qquad x = 3t^2 + t$

$t = y/2 \qquad x = 3(y/2)^2 + y/2$

$= 3y^2/4 + y/2$

$= 3y^2/4 + 2y/4$

i) $\boxed{x = y/4\,(3y+2)}$

ii) $x'(t) = 6t+1 \qquad \dfrac{dy}{dx} = \dfrac{2}{6t+1}$

$y'(t) = 2$

$(x,y) = (14,4) \qquad x = 3t^2 + t$

$y = 2t \qquad 14 = 3t^2 + t$

$4 = 2t \qquad 3t^2 + t - 14 = 0$

$t = 2 \qquad (t-2)(3t+7) = 0$

$t = 2 \quad \cancel{t = -7/3}$ ← this does not satisfy $y = 2t = 4$

$\dfrac{dy}{dt} = \dfrac{2}{6t+1} = \dfrac{2}{6(2)+1} = \dfrac{2}{13}$

$y = 2/13\, x + b$

$4 = 2/13(14) + b$

$4 = 28/13 + b$

$b = 24/13$ → $\boxed{y = 2/13\, x + 24/13}$

b) i) $r = \sin 2\theta \quad r = \cos\theta$

$\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6$

$r = \sin(\pi/3) \qquad r = \cos(\pi/6)$

$r = \sqrt{3}/2 \qquad r = \sqrt{3}/2$

Cartesian Coordinates

$x = r\cos\theta \qquad y = r\sin\theta$

$= \sqrt{3}/2 \cos(\pi/6) \qquad = \sqrt{3}/2 \sin(\pi/6)$

$= \sqrt{3}/2 \cdot \sqrt{3}/2 = 3/4 \qquad = \sqrt{3}/2 \cdot 1/2 = \sqrt{3}/4$

$\boxed{(x,y) = (3/4, \sqrt{3}/4)}$

ii) $A = \frac{1}{2}\int_0^{\pi/6} \sin^2(2\theta)\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2\theta\,d\theta$

6) a) $\lim_{x\to\infty} \frac{2+x^3}{4+5x^3} = \lim_{x\to\infty} \frac{\overset{\to 0}{2/x^3}+1}{\underset{\to 0}{4/x^3}+5} = \frac{1}{5}$ * converges to $1/5$

b) False

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$

Consider $\frac{1}{n^{1/3}}$

Positive? ✓

$\lim_{x\to\infty} \frac{1}{x^{1/3}} = 0$

Decreasing? ✓

$f'(x) = -\frac{1}{3}x^{-2/3} = \frac{-1}{x^{2/3}} < 0$

By the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ converges.

But $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^{1/3}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges because it is a p-series with $p<1$.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ converges conditionally.

d) $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$S_0 = 1/1 - 1/2$

$S_1 = 1/1 - 1/2 + 1/2 - 1/3 = 1 - 1/3$

$S_2 = 1/1 - 1/2 + 1/2 - 1/3 + 1/3 - 1/4 = 1 - 1/4$

$S_N = 1/1 - 1/N+2$ $\qquad \lim_{N\to\infty} S_N = \lim_{\to\infty} 1 - 1/N+2 = 1$

So $\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ converges to 1.

e) i) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Use integral test

$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}$ Consider $\lim_{R\to\infty} \int_2^{R} \frac{1}{x\sqrt{\ln x}}\,dx$

$u = \ln x$

$du = 1/x\,dx$.

$$\lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{1}{\sqrt{u}}\,du = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} u^{-1/2}\,du = \lim_{R\to\infty} 2u^{1/2} \Big|_{\ln 2}^{\ln R}$$

$$= \lim_{R\to\infty} 2\sqrt{\ln R} - 2\sqrt{\ln 2} = \infty$$

So $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}$ diverges, thus $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges

ii) $\sum_{n=1}^{\infty} \frac{n^7}{7^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)^7}{7^{n+1}} \cdot \frac{7^n}{n^7} \right| = \lim_{n\to\infty} \frac{(n+1)^7}{7n^7} = \frac{1}{7}$

Since $1/7 < 1$, $\sum_{n=1}^{\infty} \frac{n^7}{7^n}$ converges absolutely

iii) $\sum_{n=0}^{\infty} \frac{\cos(n)}{2+2^n}$

Consider $\sum_{n=0}^{\infty} \left| \frac{\cos(n)}{2+2^n} \right|$

$\left| \frac{\cos(n)}{2+2^n} \right| < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

Since $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $|r|<1$, it converges.

By the comparison test, $\sum_{n=0}^{\infty} \left| \frac{\cos(n)}{2+2^n} \right|$ converges. So $\sum_{n=0}^{\infty} \frac{\cos(n)}{2+2^n}$ converges absolutely.

iv) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+n^2+n+17}{3n^4+4^n+1+5}$

$$\frac{5^n+n^2+n+17}{3n^4+4^n+6} \geq \frac{5^n}{3n^4+4^n+6} \geq \frac{5^n}{4^n+4^n} = \frac{5^n}{2(4^n)}$$

for n large
$4^n \geq 3n^4+6$

$\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{2(4^n)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{5}{4}\right)^n$ since $|r|>1$, this series diverges

Thus, by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+n^2+n+17}{3n^4+4^n+1+5}$ diverges.

Alternatively, use the test for divergence!

$$\lim_{n\to\infty} \frac{5^n+n^2+n+17}{4^n+3n^4+6}$$

$$= \lim_{n\to\infty} \frac{1+\frac{n^2}{5^n}+\frac{n}{5^n}+\frac{17}{5^n}}{\frac{4^n}{5^n}+\frac{3n^4}{5^n}+\frac{6}{5^n}} = \infty$$

Alternatively alternatively, limit campare to $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{4}\right)^n$.

$$\lim_{n\to\infty} \frac{5^n+n^2+n+17}{4^n+3n^4+6} \cdot \frac{4^n}{5^n}$$

$$= \lim_{n\to\infty} \frac{20^n+4^n n^2+4^n n+4^n\cdot 17}{20^n+3\cdot 5^n n^4+6\cdot 5^n} = 1.$$

So by LCT the series diverge together.

7. a) $\frac{x^2}{1+x} = x^2 \cdot \frac{1}{1-(-x)} = x^2 \sum_{n=0}^{\infty} (-x)^n$

$$= \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$

$$= \sum_{n=2}^{\infty} (-1)^{n-2} x^n$$

b) $f(x) = \cos(x)$, $a = \pi/2$. $\quad f(a) = 0$

$f'(x) = -\sin(x) \qquad f'(a) = -1$

$f''(x) = -\cos(x) \qquad f''(a) = 0$

$f'''(x) = \sin(x) \qquad f'''(a) = 1$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(a) = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(a)}{(2n+1)!} (x-a)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left(x - \frac{\pi}{2}\right)^{2n+1}$$

8. $\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2} = \lim_{x\to\infty} \frac{1-\cos(1/x)}{1/x^2}$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{\sin(1/x)\cdot\left(-\frac{1}{x^2}\right)}{-2\,\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1}{2} \cdot \frac{\sin(1/x)}{1/x}$$

$$= \frac{1}{2}$$

By LCT both series converge.