Worksheet 3 Key

1. a) x^n converges for $-1 < x \leq 1$.
   b) n^x converges for $x \leq 0$
   c) If $\lim_{n\to\infty} b_n = \sqrt{2}$, then $\lim_{n\to\infty} b_{n-3} = \sqrt{2}$ also.
   d) S_n converges to π.
   e) If the terms of a series do not go to zero, there is no chance that the series converges. This test never tells you if a series converges
2. The terms going to zero is a necessary but not sufficient criteria for convergence. This series diverges by limit comparison to the harmonic series.
3. a) False. The second sequence is finite.
   b) True. Definition.
   c) False. See $a_n = (-1)^n$.
   d) False. Some terms could be negative. See $\sum_{n=1}^{\infty} (-1)^n$.
   e) False. If the series converges then $\lim_{n\to\infty} a_n = 0$.
   f) True. Re-index.
   g) True. Multiply partial sums by $\frac{1}{c}$.
   h) True. Only convergence on the tail matters.
   i) True. It is all zero on the tail.
4. $\sum_{n=1}^{\infty} ar^n(1-r) = \sum_{n=1}^{\infty}(ar^n - ar^{n+1})$

$$S_N = (ar - ar^2) + (ar^2 - ar^3) + \ldots + (ar^N - ar^{N+1})$$
$$= ar - ar^{N+1}$$
$$\sum_{n=1}^{\infty} (ar^n - ar^{n+1}) = \lim_{N\to\infty} S_N = ar$$

Then $\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$

5. $S_N = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \ldots + (\ln(N+1) - \ln(N))$
$= \ln(N+1) = \sum_{n=1}^{N} \ln(1 + \frac{1}{n})$

$\lim_{n \to \infty} S_N = \infty$, so the series does not converge.

6. The function is defined where the limit exists, which is only when $|x-3| < 1$. So $2 < x < 4$. By geometric series, for these values of x we have

$$f(x) = \sum_{k=1}^{\infty} (x-3)^k = \frac{(x-3)}{1-(x-3)} = \frac{x-3}{4-x}.$$

But only values of x between 2 and 4 make sense here!