Worksheet 4 Key

1. a) The test for divergence only tests for divergence. It does not test for convergence.

   b) If $\sum a_n$ converges and $0 \le b_n \le a_n$ for all n, then $\sum b_n$ converges, too. If $\sum a_n$ diverges and $0 \le a_n \le b_n$ for all n, then $\sum b_n$ diverges, too.

   c) By comparison test, $\sum x_n$ converges. No conclusion if $\sum y_n$ diverges.

   d) $\sum \frac{1}{n}$ is bigger than a convergent series, not smaller.

   e) If a_n and b_n are positive sequences and

   $$L = \lim_{n\to\infty} a_n/b_n, \text{ then}$$

   - If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ converge or diverge together.
   - If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
   - If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
   - If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
   - If $L = 0$ and $\sum a_n$ diverges, then $\sum b_n$ diverges.

2. The p-series converges if $p > 1$ and diverges otherwise.

3. $\sum_{n=0}^{\infty} ar^n$ converges if $|r| < 1$ and diverges otherwise. In a p-series, the base is $\frac{1}{n}$ and the exponent is constant. In a geometric series, the exponent is n and the base is constant.

4. a) $= 10^{10} \sum 1/n^{10}$ converges (p-series $p = 10$)

   b) $= \sum \frac{n+1}{n^{2.5}} \le \sum \frac{2n}{n^{2.5}} = 2\sum \frac{1}{n^{1.5}}$ converges

   (comparison to p-series $p = 1.5$)

   c) $\sum \frac{2}{\sqrt{n^2+2}} \ge \sum \frac{2}{\sqrt{n^2+3n^2}} = \sum \frac{1}{n}$ diverges

   (comparison to harmonic series)

   d) Test for divergence. $\lim_{n\to\infty} a_n = \frac{1}{3}$. diverges

   e) $\sum \frac{1+2^n}{2+5^n} \le \sum \frac{2^n+2^n}{5^n} = \sum 2 \cdot \left(\frac{2}{5}\right)^n$ converges.

   (comparison to geometric series, $|r| < 1$.)

f) $\sum \frac{2}{k^2+4k+3} \leq \sum \frac{2}{k^2}$ converges

(comparison to p-series, $p=2$)

g) Limit compare to $\sum 1/n^2$.

$$\lim_{n\to\infty} \frac{e^{1/n}/n}{1/n} = \lim_{x\to\infty} \frac{e^{1/x}}{1/x} \overset{LH}{=} \lim_{x\to\infty} \frac{e^{1/x}\cdot(-\frac{1}{x^2})}{-1/x^2}$$

$$= \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

So the series diverges.

h) $\sum \frac{j}{j^2-\cos^2(j)} \geq \sum \frac{j}{j^2} = \sum \frac{1}{j}$ diverges

(comparison to harmonic series.)