Worksheet 6 Key

1. a) a_n converges to $1/3$, so $\sum a_n$ diverges (TFD).
   b) The harmonic series $\sum \frac{1}{n}$.
   c) No, because if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.
   d) A series converges absolutely if $\sum |a_n|$ converges. A series converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ does not.
   e) Ratio test: If $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then
      - If $\rho < 1$, $\sum a_n$ converges absolutely.
      - If $\rho > 1$, $\sum a_n$ diverges

      Root test: If $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then
      - If $L < 1$, then $\sum a_n$ converges absolutely.
      - If $L > 1$, then $\sum a_n$ diverges.

2. a) Converges by AST
   b) $\sum_{n=1}^{\infty} (1+\frac{1}{n})^{n^2} \geq \sum_{n=1}^{\infty} (1+\frac{1}{n})^n$ which diverges by TFD.
   c) $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$. Try ratio test:

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{2^n n^2} \right| = \lim_{n\to\infty} \frac{2(n+1)}{n^2} = 0$$

   Converges absolutely.

3. a) $\sum \frac{3}{n^5+1} \leq \sum \frac{3}{n^5}$ converges by comparison to p-series $p=5$.
   b) $\sum \frac{5^n}{(11-\cos^2(n))^n} \leq \sum \frac{5^n}{10^n}$ converges by comparison to geometric series.
   c) $\sum \frac{3n}{\sqrt{2n^2+2}} \geq \sum \frac{3n}{\sqrt{2n^2+2n^2}} = \sum \frac{3n}{2n}$ diverges by comparison to constant series.

d) Converges by AST.

e) Converges by geometric, $r = \cos(5)$.

f) Try root test:

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{3n^3+2n}{4n^3+1} \right|^n} = \frac{3}{4} < 1 \quad \text{converges, abs.}$$

g) Try ratio test:

$$\lim_{n \to \infty} \left| \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} \right| = \lim_{n \to \infty} \frac{e}{n+1} = 0 < 1$$

converges, abs.

4. a) Only for alternating series $\Sigma (-1)^n a_n$.

b) True.

c) True.

d) True.

e) True.