Review for Exam 1

1) An infinite series $\sum_{n=k}^{\infty} a_n$ converges to S if partial sums converge to S

$\lim_{n \to \infty} S_n = S$

b) $\frac{1}{n}$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

diverges by p-test

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Converges

c) Geo $\sum_{n=1}^{\infty} ar^{n-1}$ converges $|r|<1$

P-test $\sum \frac{1}{n^p}$ $p>1$ converges

$p \le 1$ diverges

d) Toolbox

e) Converge Absolutely $\sum |a_n|$ converges

Conditionally $\sum |a_n|$ diverges but $\sum a_n$ converges

Yes,

No,

2. $a_n = \frac{4^n}{n!}$ $\boxed{\lim_{n \to \infty} \frac{4^n}{n!} = 0}$

$\boxed{\text{By squeeze th.}}$ $0 < \frac{4^n}{n!} < \frac{4}{1} \cdot \frac{4}{2} \cdot \frac{4}{2} \cdots \frac{4}{n} \le \frac{4}{1} \cdot \frac{4}{2} \cdot \frac{4}{3} \cdot \frac{4}{4} \left(\frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5}\right.$

$\frac{64}{6} \cdot \left(\frac{4}{5}\right)^{n-2} \to 0$ as $n \to \infty$.

b) $a_n = \sqrt{n+2} - \sqrt{n}$ $\lim_{n \to \infty} \frac{\sqrt{n+2} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} = \lim_{n \to \infty} \frac{n+2+n}{\sqrt{n+2} + \sqrt{n}}$

$= \dfrac{\lim_{n \to \infty} \frac{2n}{\sqrt{n}} + \lim_{n \to \infty} \frac{2}{\sqrt{n}}}{\lim_{n \to \infty} \left(\sqrt{1 + \frac{2}{n}} - \sqrt{1}\right)} = \dfrac{+0}{1}$ $\therefore$ Converges to zero

c) $a_n = \frac{3n^2+n+2}{5n^2+1} \Rightarrow \boxed{\lim_{n\to\infty} a_n = \frac{3}{5}}$

3. a) $\cancel{\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^n} < \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n}$ ~~geometric~~ $\cancel{\frac{1}{1-\frac{3}{5}} = \frac{5}{2}}$

$\sum_{n=1}^{\infty} \frac{3}{5}\left(\frac{3}{5}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{4}{5}\right)^{n-1}$

b) $\sum_{n=2}^{\infty}\left(\frac{1}{j} - \frac{1}{j-1}\right) = \frac{1}{2} - \frac{1}{1} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{2}} + \frac{1}{4} - \cancel{\frac{1}{3}} + \dots + \frac{1}{j} - \cancel{\frac{1}{j-1}}$

$\boxed{= \sum_{n=2}^{\infty} -1 + \frac{1}{j} = -1}$

c) $\sum_{n=1}^{\infty} \frac{3n^2+n+2}{5n^2+1}$ Test for divergence diverges
$\lim_{n\to\infty} a_n = \frac{3}{5} \neq 0$

4. a) $\sum_{k=1}^{\infty} \frac{k^{2k}}{(1+2k^2)^k}$

Root Test $\lim_{n\to\infty} \sqrt[n]{a_n} = \sqrt[k]{\left(\frac{k^2}{1+2k^2}\right)^k} = \lim_{k\to\infty} \frac{k^2}{1+2k^2} = \frac{1}{2}$

$\boxed{\frac{1}{2} < 1 \text{ so converges}}$

(b) $\sum_{n=1}^{\infty} \frac{n}{2n^4+1} < \frac{n}{2n^4} < \frac{1}{2n^4}$ Comparison test converges by p-series

(c) $\sum_{j=2}^{\infty} \frac{\sqrt{j}}{j-1} > \frac{1}{j-1}$ Comparison and limit comparison

Let $a_n = \frac{1}{j-1}$ $b_n = \frac{1}{j}$ $\lim_{n\to\infty} \frac{\frac{1}{j-1}}{\frac{1}{j}} = \frac{j}{j-1} \Rightarrow \lim_{n\to\infty} \frac{j}{j-1} = 1$ So $\frac{1}{j-1}$ diverges since $\frac{1}{j}$ diverges.

So, $\frac{\sqrt{j}}{j-1}$ diverges.

(d) $\sum_{m=2}^{\infty} \frac{(-1)^m \ln(m)}{m}$ Alternating

$b_n = \frac{\ln(m)}{m}$ which is decreasing from $2 \to \infty$

$\lim_{n\to\infty} \frac{\ln(m)}{m} \overset{L}{=} \frac{\frac{1}{m}}{1} = \frac{1}{m} \Rightarrow \lim_{m\to\infty} \frac{1}{m} = 0$

Converges.

4) e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$ $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right|$

$$= \lim_{n\to\infty} \left| \frac{n+1}{5} \right| = \infty$$

So the series $\boxed{\text{diverges}}$ by the $\boxed{\text{Ratio Test.}}$

5) a) $\displaystyle\sum_{n=0}^{\infty} (4x)^n$ $|4x| < 1$

$|x| < 1/4 \Rightarrow -1/4 < x < 1/4$

$\boxed{R = 1/4}$

Test @ $x = 1/4$

$\displaystyle\sum_{n=0}^{\infty} 1^n$ – diverges (geometric series)

Test @ $x = -1/4$

$\displaystyle\sum_{n=0}^{\infty} (-1)^n$ – diverges (geometric series)

$\boxed{\text{Interval of Convergence: } (-1/4, 1/4)}$

b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n (x-2)^n}{3^n}$ $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left| \frac{(-1)^n (x-2)^n}{3^n} \right|} = \lim_{n\to\infty} \left| \frac{2-x}{3} \right| = \left| \frac{2-x}{3} \right|$

$\left| \frac{2-x}{3} \right| < 1 \Rightarrow -1 < \frac{2-x}{3} < 1$

$-3 < 2-x < 3$

$-5 < -x < 1$

$-1 < x < 5$

$\boxed{R = 3}$

Test @ $x = -1$

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n (-3)^n}{3^n} = \sum_{n=2}^{\infty} (-1)^{2n}$ – diverges

Test @ $x = 5$

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n (3)^n}{3^n} = \sum_{n=2}^{\infty} (-1)^n$ – diverges

$\boxed{\text{Interval of Convergence: } (-1, 5)}$

c) $\sum_{n=2}^{\infty} \frac{x^n}{n!}$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$

$0 < 1$ ✓

$\boxed{R = \infty}$ Interval of Convergence: $(-\infty, \infty)$

6) $\lim_{n\to\infty} \frac{\sin(n+\pi^2)}{n^2+1} = 0$ *justification?

Squeeze $\frac{0}{n^2+1}$ and $\frac{1}{n^2+1}$

7) a) $f(x) = \frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-3x)^n$

$|3x| < 1 \Rightarrow -1 < 3x < 1$

$-1/3 < x < 1/3$

Check @ $x = 1/3$

$\sum_{n=0}^{\infty} (-1)^n$ - diverges (geometric series)

Check @ $x = -1/3$

$\sum_{n=0}^{\infty} (1)^n$ - diverges (geometric series)

$\boxed{R = 1/3}$ Interval of Convergence: $(-1/3, 1/3)$

b) $f(x) = \frac{x^3}{3+x} = \frac{x^3}{3} \cdot \frac{1}{1-(-x/3)} = \frac{x^3}{3} \sum_{n=0}^{\infty} \left(\frac{-x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{3^{n+1}}$

$\left|\frac{-x}{3}\right| < 1 \Rightarrow -1 < \frac{-x}{3} < 1$ $\boxed{R = 3}$

$-3 < -x < 3$

$-3 < x < 3$

Check @ $x = 3$

$\frac{x^3}{3} \sum_{n=0}^{\infty} (-1)^n$ — diverges (geometric series)

Check @ $x = -3$

$\frac{x^3}{3} \sum_{n=0}^{\infty} (1)^n$ - diverges (geometric series)

$\boxed{\text{Interval of Convergence: } (-3,3)}$

c) $f(x) = \frac{1}{(1+3x)^2}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

$$\frac{1}{(1+3x)^2} = \frac{1}{(1-(-3x))^2} = \sum_{n=0}^{\infty} (n+1)(-3x)^n$$

$\lim_{n\to\infty} \left| \frac{(n+2)(-3x)^{n+1}}{(n+1)(-3x)^n} \right| = |-3x| < 1$

$-1 < -3x < 1$

$1/3 > x > -1/3$

$\boxed{R = -1/3}$

Check @ $x = 1/3$

$\sum_{n=0}^{\infty} (n+1)(-1)^n$ - diverges ~~(geometric series)~~

Check @ $x = -1/3$

$\sum_{n=0}^{\infty} (n+1)(1)^n$ — diverges ~~(geometric series)~~

$\boxed{\text{Interval of convergence: } (-1/3, 1/3)}$

Use ratio test

can't use geometric because $(n+1)$ is not constant.

8) $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$

$= 1 - x^3 + x^6 - x^9 + x^{12} - \ldots$

Examine $\int_0^{1/10} 1 - x^3 + x^6 - x^9 + x^{12} - \ldots$

$= x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \frac{x^{13}}{13} - \ldots \Big|_0^{1/10}$

$= \frac{1}{10} - \frac{1}{4 \cdot 10^4} + \frac{1}{7 \cdot 10^7} - \frac{1}{10 \cdot 10^{10}} + \frac{1}{13 \cdot 10^{13}} - \ldots$

$= \sum_{n=0}^{\infty} \frac{1}{(3n+1)} \cdot \frac{1}{10^{3n+1}} \cdot (-1)^n$

$|S - S_N| < a_{N+1} = \frac{1}{3(N+1)+1} \cdot \frac{1}{10^{3(N+1)+1}} \leq 10^{-11}$

$= \frac{1}{3N+4} \cdot \frac{1}{10^{3N+4}} \leq 10^{-11}$

$10^{3N+4}(3N+4) \geq 10^{11}$

$\left\{ \begin{matrix} 10^{3(2)+4}(3(2)+4) \geq 10^{11} \\ 10^{10}(10) \geq 10^{11} \end{matrix} \right\}$

$N = \cancel{2}\ 3$

$S_2 = \frac{1}{10} - \frac{1}{4 \cdot 10^4} + \frac{1}{7 \cdot 10^7} = 0.0999$

But what is S?

$\int_0^{1/10} \frac{1}{1+x^3}\, dx = \int_0^{1/10} \sum_{n=0}^{\infty} (-x^3)^n\, dx$

$= \int_0^{1/10} \sum_{n=0}^{\infty} (-1)^n x^{3n}\, dx$

$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \right]_0^{1/10}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(3n+1)\, 10^{3n+1}}$

9. a) True. $\{a_n\}$ is a strictly increasing sequence and is bounded above, so the sequence converges by the monotone convergence theorem.

b) FALSE! FALSE! FALSE! $\lim_{n\to\infty} a_n = 0$ is a necessary but insufficient condition for $\sum_{n=1}^{\infty} a_n$ to converge.

c) True, because $\sqrt{n^3+n} \geq \sqrt{n^3}$.

d) False. Absolutely convergent series converge, but some series converge only conditionally.

e) True. This is how convergence of a series is defined. A series converges if its sequence of partial sums converges.

f) False! $\lim_{n\to\infty} a_n = 0$.

g) False. The series must converge on $(0,2)$, but we have to check the endpoints.

h) False. $\sum a_n$ is smaller than a divergent series, which tells us nothing.

i) False. $\sum b_n$ is larger than a convergent series, which also tells us nothing.

j) False. Counterexample $a_n = 1$, $b_n = -1$.

k) True. Say $\sum_{n=1}^{\infty} a_n = a$. Then $S_N = \sum_{n=1}^{N} (a_n + b_n)$

$$= \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n,$$

and $\lim_{N\to\infty} S_N = a + \lim_{N\to\infty} \sum_{n=1}^{N} b_n$ does not converge.