Worksheet 9. Key

1\. a) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

b) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

c) $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$

$f'(x) = 2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3$

$f''(x) = 3 \cdot 2 + 4 \cdot 3 \cdot 2x + 5 \cdot 4 \cdot 3x^2$

$f^{(3)}(x) = 4! + 5!x$

$f^{(4)}(x) = 5!$

$f^{(j)}(x) = 0$ for $j > 4$.

Then $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + 2x + \frac{3!}{2!}x^2 + \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4$

$= 1 + 2x + 3x^2 + 4x^3 + 5x^4$

d) $f(x) = 1 + 2x + 3x^2 + 4x^3$

$f'(x) = 2 + 3!x + 4 \cdot 3x^2$

$f''(x) = 3! + 4!x$

$f^{(3)}(x) = 4!$

$f^{(j)}(x) = 0$ for $j > 0$.

$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$

$= 10 + 20(x-1) + 15(x-1)^2 + 4(x-1)^3$

$= 1 + 2x + 3x^2 + 4x^3$

2\. $\cos(x) = \frac{d}{dx} \sin(x) = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ Ratio Test.

$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+1)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{2n+1} \right| = 0 < 1.$

So the series converges absolutely for all $x \in \mathbb{R}$.

3. a) $f(x) = \ln(1+x)$

$f'(x) = (1+x)^{-1}$

$f''(x) = -1\,(1+x)^{-2}$

$f^{(3)}(x) = (-1)^2 \cdot 2!\,(1+x)^{-3}$

$f^{(4)}(x) = (-1)^3 \cdot 3!\,(1+x)^{-4}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n} \Rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!} x^n + f(0)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

Ratio test : Converges on $(-1,1)$.

b) $g(x) = e^{2x}$

$g'(x) = 2e^{2x}$

$g''(x) = 2^2 e^{2x}$

$g^{(n)}(x) = 2^n e^{2x}$ $\quad g^{(n)}(0) = 2^n$

$$xe^{2x} = x\,g(x) = x \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1}$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{(n-1)!} x^n$$

Ratio test: $\lim_{n\to\infty} \left| \frac{2^n x^{n+1}}{n!} \cdot \frac{(n-1)!}{2^{n-1} x^n} \right| = \lim_{n\to\infty} \left| \frac{2x}{n} \right| = 0 < 1.$

The series converges on $(-\infty, \infty)$.

4. Use $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$,

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, and $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

a) $$\frac{x^2}{1-3x} = x^2 \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^{n+2}$$

$$= \sum_{n=2}^{\infty} 3^{n-2} x^n.$$ Converges for $|x| < 1/3$.

b) $$e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} x^n$$

c) $$x e^{2x} = x \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{2^{n-1} x^n}{(n-1)!}$$

d) $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

e) $$x^5 \sin(3x^2) = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{4n+7}$$

f) $$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n}}{(2n)!\, 2} x^{2n}$$

$$= \frac{1}{2} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n}}{(2n)!\, 2} x^{2n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n}$$