- The Nth **partial sum** of the infinite series $\sum_{n=1}^{\infty} a_n$ is $S_N = \sum_{n=1}^{N} a_n$. A series converges if and only if its sequence of partial sums $\{S_N\}$ converges.

- **Test for Divergence (The Hammer):** If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. That is to say, if the terms of the series are not going to zero, then the series has no chance to converge.

  Remember that the converse is not true. That is, if $\lim_{n\to\infty} a_n = 0$ then the series only has a *chance* to converge, and another test must be used.

- A **geometric series** of the form $\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$ will diverge if $|r| \geq 1$, by the test for divergence. But if $|r| < 1$, the series will converge and

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

  Remember that a is the first term of the sequence, and r is the common ratio between terms, so both can be easily computed.

- **Integral Test:** Suppose $f(x)$ is continuous, positive, and monotone decreasing on $[1, \infty)$ and $f(n) = a_n$ for all positive integers n from some point on. Then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_{n=1}^{\infty} f(x)\ dx$ converge or diverge together.

- A **p-series** of the form $\sum_{n=1}^{\infty} 1/n^p$ will converge if $p > 1$ and diverge if $p \leq 1$. Note that the special case for $p = 1$ is the Harmonic Series, which famously diverges.

- **Comparison Test:** Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
  - If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all large n, then $\sum_{n=1}^{\infty} a_n$ also converges.
  - If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \geq b_n$ for all large n, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **Limit Comparison Test:** Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive term series and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

  - If $0 < L < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.
  - If $\sum_{n=1}^{\infty} b_n$ converges and $L = 0$, then $\sum_{n=1}^{\infty} a_n$ also converges.
  - If $\sum_{n=1}^{\infty} b_n$ diverges and $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ also diverges.

- **Alternating Series Test:** If $\{a_n\}$ is a positive, monotone decreasing sequence, and $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

- **Alternating Series Estimation Theorem:** If the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges to S, then

$$|S - S_N| < a_{N+1},$$

  where S_N is the Nth partial sum of the series. That is to say, the error in the Nth partial sum is less than the $(N+1)$st term of the series.

- To show that a sequence $\{a_n\}$ is **monotone decreasing**, one has two options:
  - Algebraically prove that $a_n \geq a_{n+1}$ for all large n.
  - Find a differentiable function $f(x)$ such that $f(n) = a_n$ for all positive integers n from some point on, and show that $f'(x) < 0$ for all large x.
- **Ratio Test:** Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.
  - If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $L = 1$, then the test is inconclusive.
- **Root Test:** Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.
  - If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
  - If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  - If $L = 1$, then the test is inconclusive.
- **Telescoping series** are positive term series in which the general partial sum is easy to compute due to cancellation between the terms. This is best demonstrated with an example. Suppose we wish to investigate the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

First we can use partial fraction decomposition to rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Then the first few partial sums are:

$$S_1 = \left(\frac{1}{1} - \frac{1}{2}\right) = 1 - \frac{1}{2}$$
$$S_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$
$$S_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

In general, the Nth partial sum is

$$S_N = 1 - \frac{1}{N+1}.$$

Since a series converges if and only if its sequence of partial sums converges, and

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$$

we conclude that the series converges to 1.